$$\int\left(\frac{3}{x-1}+\frac{1}{x^2+x+1}\right)dx$$

$$=3\ln|x-1|+\int\left[\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right]dx \qquad x^2+x+1=x^2+x+\frac{1}{4}+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$=3\ln|x-1|+\int\left[\frac{\frac{4}{3}}{\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1}\right]dx \qquad \text{Multiply top and bottom by } \frac{4}{3}.$$

$$=3\ln|x-1|+\int\left[\frac{\frac{4}{3}}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}\right]dx \qquad \frac{4}{3}\left(x+\frac{1}{2}\right)^2=\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2=\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2$$

$$=3\ln|x-1|+\left(\frac{4}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\int\frac{du}{u^2+1} \qquad u=\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}},\quad du=\frac{2}{\sqrt{3}}dx$$

$$=3\ln|x-1|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C \qquad \int\frac{du}{u^2+1}=\tan^{-1}u+C$$

7.2 Exercises

1–9 Use the guidelines discussed in this section to write the form of the partial fraction decomposition of the given rational function. Do not solve for the coefficients in your decomposition.

1. $\dfrac{2}{x(x-1)}$

2. $\dfrac{5x}{(x+4)(x-2)(x+7)}$

3. $\dfrac{2x+5}{(x+1)(x-3)^3}$

4. $\dfrac{3x-1}{(2x+3)(x^2+2)}$

5. $\dfrac{x-4}{(x^2+x+2)^3}$

6. $\dfrac{14x-3}{(3x-1)(x^2+1)^2}$

7. $\dfrac{3}{2x^2-5x-3}$

8. $\dfrac{2x+1}{5x^3-11x^2+7x-1}$

9. $\dfrac{4x^2-1}{x^5+x^4+2x^3+2x^2+x+1}$

10–35 Use the partial fractions method to evaluate the given integral.

10. $\displaystyle\int\frac{dx}{x(x+2)}$

11. $\displaystyle\int\frac{4}{(3x-1)x}dx$

12. $\displaystyle\int\frac{x-4}{(x+2)(x-1)}dx$

13. $\displaystyle\int\frac{2}{(x+3)(2x+5)}dx$

14. $\displaystyle\int\frac{5x+2}{(x+1)(3x-1)(x+3)}dx$

15. $\displaystyle\int\frac{2x}{(x-2)^2(x+1)}dx$

16. $\displaystyle\int\frac{x}{(x+2)^3}dx$

17. $\displaystyle\int\frac{2-z}{z(z+1)^2}dz$

18. $\displaystyle\int\frac{dx}{x(x^2+2)}$

19. $\displaystyle\int\frac{dx}{x^2+3x+2}$

20. $\displaystyle\int\frac{2-z}{z^2-1}dz$

21. $\displaystyle\int\frac{2}{x-x^3}dx$

22. $\displaystyle\int\frac{s-3}{s(s-1)(s+3)}ds$

23. $\displaystyle\int\frac{t-1}{t^3+t^2+t+1}dt$

24. $\displaystyle\int\frac{x^2+1}{(x^2+2x+3)^2}dx$

25. $\displaystyle\int\frac{x^2-9}{x^4+3x^3}dx$

26. $\displaystyle\int\frac{5x^3-5x-40}{x^4+x^3+4x^2+4x}dx$

27. $\displaystyle\int\frac{11x-12}{x(x^2+x-6)}dx$

28. $\displaystyle\int\frac{3x^4-5x^3+15x^2-8x+20}{x^5-2x^4+4x^3-8x^2+4x-8}dx$

29. $\displaystyle\int \frac{32}{x^4-4x^3-2x^2+12x+9}\,dx$

30. $\displaystyle\int \frac{2x^4+4x^2-x+2}{\left(1+x^2\right)^3}\,dx$

31. $\displaystyle\int \frac{16v}{v^4-16}\,dv$

32. $\displaystyle\int \frac{-x^2-1}{x^4+5x^2+6}\,dx$

33. $\displaystyle\int \frac{3x}{x^3-1}\,dx$

34. $\displaystyle\int \frac{x}{(x+a)(x-b)}\,dx$

35. $\displaystyle\int \frac{a}{x\left(x^2+b\right)}\,dx$

36–41 Use the Heaviside cover-up method to evaluate the given integral.

36. $\displaystyle\int \frac{6x^2-19x-12}{x\left(x^2-x-6\right)}\,dx$

37. $\displaystyle\int \frac{16(x+2)}{(x-7)\left(x^2-1\right)}\,dx$

38. $\displaystyle\int \frac{2v^2+13v+6}{v^3-v^2-10v-8}\,dv$

39. $\displaystyle\int_0^1 \frac{-5t^2+8t+19}{t^3+2t^2-5t-6}\,dt$

40. $\displaystyle\int_1^2 \frac{6s^3-38s^2+48s+12}{s^4-6s^3+5s^2+12s}\,ds$

41. $\displaystyle\int_{-1}^1 \frac{-x^3+4x^2-11x-6}{\left(x^2-4\right)\left(x^2-9\right)}\,dx$

42. If $k \in \mathbb{R}$, use partial fractions to prove the formula

$$\int \frac{1}{x^2-k^2}\,dx = -\frac{1}{2k}\ln\left|\frac{x+k}{x-k}\right| + C.$$

43. Complete the square in the denominator of Example 1 and use the formula in Exercise 42 to arrive at the same answer as in Example 1.

44–46 Use your approach taken in Exercise 42 to establish the given formula for $k, l \in \mathbb{R}$.

44. $\displaystyle\int \frac{dx}{x(kx+l)} = \frac{1}{l}\ln\left|\frac{x}{kx+l}\right| + C$

45. $\displaystyle\int \frac{dx}{x^2(kx+l)} = -\frac{1}{lx} - \frac{k}{l^2}\ln\left|\frac{x}{kx+l}\right| + C$

46. $\displaystyle\int \frac{x}{(kx+l)^2}\,dx = \frac{1}{k^2}\left(\frac{l}{kx+l} + \ln|kx+l|\right) + C$

47–61 Use any of the techniques seen in Examples 3–7 to evaluate the given integral (definite or indefinite, as indicated). Whenever applicable, use the formulas from Exercises 42, 44–46. (**Note:** If the integrand is not a proper rational function, be sure to divide first.)

47. $\displaystyle\int \frac{x^2+1}{x^3+1}\,dx$

48. $\displaystyle\int \frac{3x^3-4x^2+2}{x^2-x}\,dx$

49. $\displaystyle\int_5^8 \frac{dx}{x^2-16}$

50. $\displaystyle\int \frac{2}{s\left(s^2+1\right)^2}\,ds$

51. $\displaystyle\int_1^2 \frac{dx}{x(x+2)}$

52. $\displaystyle\int_0^{\sqrt{7}} \frac{x^3}{x^2+9}\,dx$

53. $\displaystyle\int \frac{3x^3}{x^3-1}\,dx$

54. $\displaystyle\int_1^2 \frac{dx}{x^2(x+2)}$

55. $\displaystyle\int \frac{3x^2-8x+2}{(x-3)\left(x^2-2x+2\right)}\,dx$

56. $\displaystyle\int_0^2 \frac{x^2-5x}{x^2+x+3}\,dx$

57. $\displaystyle\int_0^3 \frac{v}{(v+5)^2}\,dv$

58. $\displaystyle\int \frac{16x^5}{(x-1)^2(3-x)^4}\,dx$

59. $\displaystyle\int \frac{a}{(x-b)(x-c)}\,dx$

60. $\displaystyle\int \frac{z^2+8z+9}{\left(z^2+2z+3\right)^2}\,dz$

61. $\displaystyle\int \frac{x^4+1}{x^3+4x}\,dx$

62–73 Combine integration by substitution and the partial fractions method to evaluate the given integral. (When applicable, also use integration by parts.)

62. $\displaystyle\int \frac{\cos x}{\sin^2 x + 2\sin x}\,dx$

63. $\displaystyle\int \frac{\sin x \cos x}{(\cos x - 1)(\cos x + 2)}\,dx$

64. $\displaystyle\int \frac{2\sec^2 x}{\tan^2 x - 1}\,dx$

65. $\displaystyle\int \frac{6}{x\left[(\ln x)^2 - \ln\left(x^3\right)\right]}\,dx$

66. $\displaystyle\int \frac{4e^x}{e^{2x}+2e^x-3}\,dx$

67. $\displaystyle\int \frac{\sqrt{x}}{\left(1+\sqrt{x}\right)^2}\,dx$

68. $\displaystyle\int \frac{\sqrt[3]{x}}{\left(1-\sqrt[3]{x}\right)^2}\,dx$

69. $\displaystyle\int \frac{dx}{x\sqrt{1+\sqrt{x}}}$

70. $\displaystyle\int e^x \ln\left(e^{2x}+2\right)dx$

71. $\displaystyle\int \frac{\sin(\ln x)}{x\left[\cos^2(\ln x) + \cos(\ln x)\right]}\,dx$

72. $\displaystyle\int \frac{e^{(3/2)x}}{e^x+1}\,dx$

73. $\displaystyle\int \frac{\left[\ln\left(w^2\right)\right]\left[\arctan(\ln w)\right]}{w}\,dw$

74. Use the disk method to find the volume of the solid generated by revolving the graph of $f(x) = 2/\sqrt{x^2 - 3x - 10}$, $6 \le x \le 10$, about the x-axis.

75. Use the shell method to find the volume of the solid obtained by revolving the region bounded by $g(x) = 1/(-x^2 + 2x + 8)$, $y = 0$, $x = 0$, and $x = 2$ about the line $x = -1$.

76. Find the centroid of the region bounded by the graphs of $f(x) = \dfrac{-4x - 1}{x^2 - x - 2}$, $y = 0$, $x = 0$, and $x = 1$.

77.* Suppose that we are looking for a function $y = y(t)$ whose rate of change $y'(t)$ is directly proportional to $(c^4 - y^4)/y^2$ (c is a constant). In other words, $y(t)$ then satisfies the equation $\dfrac{dy}{dt} = m(c^4 - y^4)/y^2$ for some constant m. (Such an equation, containing a derivative of an unknown function, is called a differential equation. You will learn more about differential equations in Chapter 8. The equation in this exercise is used in physical chemistry.) Use partial fractions to find an implicit formula for $y(t)$. (**Hint:** As a first step, rewrite the equation in differential form, $y^2/(c^4 - y^4)\,dy = m\,dt$, and then use partial fractions to integrate.)

78.* If the ability of the environment to support a population is limited and, thus, the population cannot grow larger than a certain size, the model $P'(t) = mP(t)$ and its solution $P(t) = P(0)e^{mt}$ is no longer adequate to describe the population growth. Instead, the so-called *logistic model* $dP/dt = mP(L - P)$ has been proposed, where L is the upper limit of the population size. If the world's population in 1940 was 2.3 billion, which grew to 6.9 billion by 2010, and supposing that Earth cannot support more than $L = 15$ billion people, what will be the world's population by 2050? (See the hint given in Exercise 77.)

79.* Assuming that the rate at which a disease is spreading after an infected person enters a community of N susceptible people is proportional to the product of the number of already infected individuals by the number of still-healthy people, and letting $I(t)$ stand for the number of individuals already infected, this latter function satisfies $dI/dt = kI(N - I)$, with $I(0) = 1$. Use your approach taken in the previous two exercises to find a formula for $I(t)$.

7.2 Technology Exercises

80–81 Use a computer algebra system to find the partial fraction decomposition of the given rational function.

80. $f(x) = \dfrac{3x^7 + 20x^6 + 81x^5 + 123x^4 - 61x^3 - 1033x^2 - 2056x - 2401}{x^8 + 7x^7 + 31x^6 + 66x^5 + 78x^4 - 78x^3 - 203x^2 - 245x + 343}$

81. $f(x) = \dfrac{3x^8 + 17x^7 + 47x^6 + 97x^5 + 156x^4 + 207x^3 + 194x^2 + 107x + 42}{x^6 + 6x^5 + 16x^4 + 26x^3 + 27x^2 + 20x + 12}$