7.1 Exercises

1–4 Evaluate the integral using integration by parts with the suggested choice for u and dv.

1. $\int xe^x\,dx;\quad u=x,\quad dv=e^x\,dx$
2. $\int 4x\cos 2x\,dx;\quad u=x,\quad dv=2\cos 2x\,dx$
3. $\int 4x^3\ln x\,dx;\quad u=\ln x,\quad dv=4x^3\,dx$
4. $\int \arctan x\,dx;\quad u=\arctan x,\quad dv=dx$

5–36 Evaluate the integral. (**Hint:** For some integrals, you may use an alternative method of integration in addition to or instead of integration by parts.)

5. $\int (t+1)e^{5t}\,dt$
6. $\int x^2e^{x^3}\,dx$
7. $\int 2x\sin x\,dx$
8. $\int \arcsin x\,dx$
9. $\int x\ln\left(x^2\right)dx$
10. $\int \frac{dx}{x\ln\left(x^2\right)}$
11. $\int x^2\ln x\,dx$
12. $\int \frac{\ln\left(x^2\right)}{x}\,dx$
13. $\int (s-3)e^{s-3}\,ds$
14. $\int (s-3)e^{(s-3)^2}\,ds$
15. $\int (2x+7)e^{3x-1}\,dx$
16. $\int \sqrt{z}\ln\left(z^2\right)dz$
17. $\int x\sqrt{x-2}\,dx$
18. $\int (3x+1)\sqrt[3]{x-1}\,dx$
19. $\int \sqrt[3]{3x}\ln x\,dx$
20. $\int \log_3 x\,dx$
21. $\int \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$
22. $\int (\theta+1)\sin\theta\,d\theta$
23. $\int x\sec x\tan x\,dx$
24. $\int \sec^2 x\tan x\,dx$
25. $\int x\csc^2 x\,dx$
26. $\int x\csc^2\left(x^2\right)dx$
27. $\int \frac{3x-1}{e^x}\,dx$
28. $\int x^{-2}e^{1/x}\,dx$
29. $\int (t+2)\cosh t\,dt$
30. $\int \frac{x}{\sqrt{2x-5}}\,dx$
31. $\int x\sqrt{x+1}\,dx$
32. $\int (3-4x)\sinh 2x\,dx$
33. $\int (3x+2)\operatorname{sech}^2 x\,dx$
34. $\int 2\operatorname{csch}^2 x\coth x\,dx$
35. $\int x\sqrt{2x-5}\,dx$
36. $\int \theta\cos\left(\theta^2\right)d\theta$

37–48 Evaluate the integral. If necessary, use integration by parts more than once.

37. $\int \theta^2\cos\theta\,d\theta$
38. $\int t^3e^t\,dt$
39. $\int e^x\cos x\,dx$
40. $\int \sin x\cos x\,dx$
41. $\int 9t^2\sin 3t\,dt$
42. $\int 2t^2e^{5t+1}\,dt$
43. $\int e^{3x}\sin 2x\,dx$
44. $\int 2e^x\sin x\cos x\,dx$
45. $\int x^2\sinh x\,dx$
46. $\int \sec^3 x\,dx$
47. $\int \cos(\ln x)\,dx$
48. $\int (\ln x)^3\,dx$

49–56 Combine the method of integration by parts with substitution to evaluate the integral.

49. $\int e^{\sqrt{x}}\,dx$
50. $\int 2x^3\sin\left(x^2+1\right)dx$
51. $\int \frac{\arctan\left(1/\sqrt{x}\right)}{2\sqrt{x}}\,dx$
52. $\int t\ln(2-t)\,dt$
53. $\int \frac{\arccos\sqrt{x}}{\sqrt{x}}\,dx$
54. $\int \sin(2x)e^{\sin x}\,dx$
55. $\int 9x^2(\ln x)^2\,dx$
56. $\int \frac{\cos(1/\theta)}{\theta^3}\,d\theta$

57–64 Evaluate the definite integral. (Use integration by parts only when necessary.)

57. $\int_{1/2}^{1}\arccos x\,dx$
58. $\int_{2}^{2e}\ln\frac{x}{2}\,dx$
59. $\int_0^{\pi}e^{2\theta}\sin\theta\,d\theta$
60. $\int_0^{\pi}t^2\sin t\,dt$
61. $\int_{\sqrt{3}/3}^{\sqrt{3}}\arctan\frac{1}{x}\,dx$
62. $\int_0^1 x^3e^x\,dx$
63. $\int_0^1\ln\left(t^2+1\right)dt$
64. $\int_1^e\frac{\ln x^2}{x^2}\,dx$

65–70 Integration by parts can often be used to evaluate integrals involving inverses of functions, and in fact leads to a general formula, as follows.

$$\int f^{-1}(x)dx=\int yf'(y)dy$$

Let $y=f^{-1}(x)$, so $x=f(y)$ and $dx=f'(y)dy$.

$$=yf(y)-\int f(y)dy$$

$u=y \quad dv=f'(y)dy$
$du=dy \quad v=f(y)$

$$=xf^{-1}(x)-\int f(y)dy$$

For instance, if we let $f(x)=e^x$, then $y=f^{-1}(x)=\ln x$.

$$\begin{aligned}\int \ln x\,dx&=\int f^{-1}(x)dx\\&=xf^{-1}(x)-\int e^y\,dy\\&=x\ln x-e^y\\&=x\ln x-e^{\ln x}\qquad y=\ln x\\&=x\ln x-x\end{aligned}$$

In Exercises 65–70, use this method to evaluate the given indefinite integral. (**Hint:** In starred exercises, show that $\cosh\left(\sinh^{-1}x\right)=\sqrt{1+x^2}$ and that $\cosh\left(\tanh^{-1}x\right)=1/\sqrt{1-x^2}$ as part of the process toward the answers.)

65. $\int \sin^{-1}x\,dx$

66. $\int \cos^{-1}x\,dx$

67. $\int \tan^{-1}x\,dx$

68. $\int \log_2 x\,dx$

69.* $\int \sinh^{-1}x\,dx$

70.* $\int \tanh^{-1}x\,dx$

71. Use integration by parts to find the area of the region bounded by the graphs of $y=6\tan^{-1}(2x)$, $y=0$, and $x=\sqrt{3}/2$.

72. Use integration by parts to find the area of the region bounded by the graphs of $y=\sin(\ln x)$ and $y=0$ $(1\le x\le e^{\pi})$.

73. Consider the region bounded by the graphs of $y=e^{-x}$, $y=0$, $x=0$, and $x=1$.

a. Find the centroid of the region.

b. Use the shell method to find the volume of the solid generated by revolving the region about the y-axis.

74. Repeat Exercise 73 for the region bounded by the graphs of $y=x\cos x$ and $y=0$ $(0\le x\le \pi/2)$.

75. Consider the region bounded by the graphs of $y=\arcsin x$, $x=0$, and $y=\pi/2$.

a. Find the centroid of the region.

b. Rotate the region about the x-axis and use the shell method to find the volume of the resulting solid.

76. Use the shell method to find the volume of the solid obtained by revolving the region bounded by the graphs of $y=2^x$, $y=0$, $x=0$, and $x=1$ about the line $x=-1$.

77. Find the centroid of the region bounded by the graphs of $y=\ln x$, $y=0$, and $x=e$.

78. The definite integral $\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin(nx)\,dx$, $n\in\mathbb{N}$ is called a *Fourier coefficient.* Use integration by parts to verify that its value is $(-1)^{n+1}\frac{2}{n}$. (The theory of Fourier series is very important in applied mathematics. You will be introduced to infinite series of functions in Chapter 10.)

79. Use integration by parts to prove that if $f(x)$ is continuously differentiable on $[-\pi,\pi]$, then the limit of the Fourier coefficients is 0.

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx=0$$

80. Use integration by parts to prove the formula

$$\int x^n\ln x\,dx=\frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x-1\right]+C.$$

81–87 Use integration by parts to prove the given reduction formula for $n\in\mathbb{N}$.

81. $\int \sin^n x\,dx=-\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int \sin^{n-2}x\,dx$

82. $\int x^n\cos(kx)\,dx=\frac{x^n}{k}\sin(kx)-\frac{n}{k}\int x^{n-1}\sin(kx)\,dx$

83. $\int x^n\sin(kx)\,dx=-\frac{x^n}{k}\cos(kx)+\frac{n}{k}\int x^{n-1}\cos(kx)\,dx$

84. $\int x^ne^{kx}\,dx=\frac{1}{k}x^ne^{kx}-\frac{n}{k}\int x^{n-1}e^{kx}\,dx,\quad k\neq 0$

85.* $\int \tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int \tan^{n-2}x\,dx,\quad n\neq 1$

86. $\int \sec^n x\,dx=\frac{\tan x\sec^{n-2}x}{n-1}+\frac{n-2}{n-1}\int \sec^{n-2}x\,dx,\ n\neq 1$

87. $\int (\ln x)^n\,dx=x(\ln x)^n-n\int(\ln x)^{n-1}\,dx$

88–95 Use the above reduction formulas to evaluate the integrals.

88. $\int \sin^5 x\,dx$

89. $\int x^3\cos 2x\,dx$

90. $\int x^3\sin(\pi x)\,dx$

91. $\int \cos x\sin^3 xe^{\sin x}\,dx$

92. $\int \sec^6 x\,dx$

93. $\int (\ln x)^3\,dx$

94. $\int \tan^4 x\,dx$

95. $\int_0^1 x^4e^x\,dx$

96. Use the appropriate reduction formula to evaluate the definite integral $\int_0^{\pi/2}\sin^6 x\,dx$. Can you conjecture a possible formula for $n=8, 10, 12, \ldots$?

97. Use integration by parts to prove the formula

$$\int \sin(kx)e^{lx}\,dx=\frac{l\sin(kx)-k\cos(kx)}{k^2+l^2}e^{lx}+C.$$

98. Use your solution to Exercise 97 to find a similar formula for $\int \cos(kx)e^{lx}\,dx$.

99–102 In Exercise 91, you had to use a reduction formula to evaluate $\int u^3 e^u \, du$. Note that you can obtain the same answer by using the *method of undetermined coefficients*, as follows. Assuming that the answer has the form

$$Au^3e^u + Bu^2e^u + Cue^u + De^u + E,$$

differentiating yields

$$u^3e^u = Au^3e^u + (3A+B)u^2e^u + (2B+C)ue^u + (C+D)e^u.$$

By equating coefficients we obtain

$$A = 1, \quad 3A + B = 0, \quad 2B + C = 0, \quad \text{and} \quad C + D = 0.$$

Solving the above system yields $B = -3$, $C = 6$, and $D = -6$ (while $E = C_1$ is arbitrary).

In Exercises 99–102, use the method of undetermined coefficients to evaluate the given integral. (Note that this method will become important in Section 7.2.)

99. $\int 8x^3 e^{2x} \, dx$

100. $\int (x^4 - x) e^x \, dx$

101. $\int 13e^{3x} (\sin 2x) \, dx$ (**Hint:** Anticipate the answer in the form stated below.)

$$Ae^{3x} \sin 2x + Be^{3x} \cos 2x$$

102. $\int 5 \sin 2x \cos 3x \, dx$ (**Hint:** Anticipate the answer in the form stated below.)

$$A \sin 2x \cos 3x + B \cos 2x \cos 3x$$
$$+ C \cos 2x \sin 3x + D \sin 2x \sin 3x + E$$