b. $\displaystyle\int_0^{1/3}\frac{dx}{1-x^2}=\left[\tanh^{-1}x\right]_0^{1/3}=\tanh^{-1}\left(\frac{1}{3}\right)$

Note that we used $\tanh^{-1}x$ as the antiderivative (as opposed to $\coth^{-1}x$) because, in the given interval of integration, $x^2<k^2$ $(k=1)$. Further, if we apply the result of Exercise 74, we can rewrite our answer as follows.

$$\tanh^{-1}\left(\frac{1}{3}\right)=\frac{1}{2}\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)=\frac{1}{2}\ln 2$$

6.6 Exercises

1–4 Find the value of the function.

1. **a.** $\cosh 0$ **b.** $\sinh(-2)$ **c.** $\sinh\frac{\pi}{2}$

2. **a.** $\tanh 0$ **b.** $\cosh 1$ **c.** $\sinh(\ln 3)$

3. **a.** $\sinh^{-1}0$ **b.** $\cosh^{-1}1$ **c.** $\sinh^{-1}2$

4. **a.** $\tanh^{-1}0$ **b.** $\operatorname{sech}^{-1}1$ **c.** $\operatorname{csch}^{-1}(-3)$

5–10 Use the given equation to classify the hyperbolic function as even or odd. Then use the definition of the function to prove your assertion.

5. $\sinh(-x)=-\sinh x$

6. $\cosh(-x)=\cosh x$

7. $\tanh(-x)=-\tanh x$

8. $\coth(-x)=-\coth x$

9. $\operatorname{sech}(-x)=\operatorname{sech}x$

10. $\operatorname{csch}(-x)=-\operatorname{csch}x$

11–22 Sketch the graph of the equation on a piece of paper. (**Hint:** Study the graphs of the six hyperbolic functions and their inverses in the text.)

11. $y=1-\sinh x$

12. $y=\cosh 2x-3$

13. $y=-\tanh(x-1)$

14. $y=2-2\operatorname{sech}(x-1)$

15. $y=-\frac{1}{2}\operatorname{csch}x+1$

16. $y=-\frac{1}{4}\coth x-2$

17. $y=\frac{1}{3}\sinh^{-1}(x-3)$

18. $y=2\cosh^{-1}(1-x)$

19. $y=-\tanh^{-1}\left(\frac{x}{3}\right)$

20. $y=-3\operatorname{csch}^{-1}(x+2)$

21. $y=\operatorname{sech}^{-1}\left(1-\frac{x}{2}\right)-1$

22. $y=2\coth^{-1}\left(\frac{x}{2}+1\right)$

23–42 Verify the given identity.

23. $e^{kx}=\cosh kx+\sinh kx$

24. $e^{-kx}=\cosh kx-\sinh kx$

25. $\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$

26. $\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y$

27. $\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$

28. $\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y$

29. $\sinh 2x=2\sinh x\cosh x$

30. $\cosh 2x=\cosh^2 x+\sinh^2 x$

31. $\cosh^2 x=\dfrac{\cosh 2x+1}{2}$

32. $\coth^2 x=1+\operatorname{csch}^2 x$

33. $\tanh(x+y)=\dfrac{\tanh x+\tanh y}{1+\tanh x\tanh y}$

34. $\tanh(x-y)=\dfrac{\tanh x-\tanh y}{1-\tanh x\tanh y}$

35. $\tanh 2x=\dfrac{2\tanh x}{1+\tanh^2 x}$

36. $\coth 2x=\dfrac{1+\coth^2 x}{2\coth x}$

37. $\sinh\dfrac{x}{2}=\pm\sqrt{\dfrac{\cosh x-1}{2}}$

38. $\cosh\dfrac{x}{2}=\sqrt{\dfrac{\cosh x+1}{2}}$

39. $\tanh\dfrac{x}{2}=\pm\sqrt{\dfrac{\cosh x-1}{\cosh x+1}}$

40. $(\cosh x+\sinh x)^2=\cosh 2x+\sinh 2x$

41. $(\cosh x+\sinh x)^n=\cosh nx+\sinh nx,\quad n\in\mathbb{N}$

42. $(\cosh x-\sinh x)^n=\cosh nx-\sinh nx,\quad n\in\mathbb{N}$

43–50 Verify the differentiation or integration formula.

43. $\frac{d}{dx}(\sinh x) = \cosh x$

44. $\int \sinh x\,dx = \cosh x + C$

45. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

46. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

47. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

48. $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

49. $\int \tanh x\,dx = \ln(\cosh x) + C$

50. $\int \coth x\,dx = \ln|\sinh x| + C$

51–62 Find the given derivative.

51. $\frac{d}{dx}\left(\sinh^2 x\right)$

52. $\frac{d}{dt}\left(\operatorname{sech}^2 2t\right)$

53. $\frac{d}{dx}\left(\tanh^2 x\right)$

54. $\frac{d}{dx}\left(\cosh^2 \frac{2x+5}{3}\right)$

55. $\frac{d}{dx}\left(e^{-x}\sinh 2x\right)$

56. $\frac{d}{dx}\left[\operatorname{sech}\left(1-x^2\right)\right]$

57. $\frac{d}{dx}\sqrt{1+\tanh 2x}$

58. $\frac{d}{dx}\left[\ln\left(\cosh(2x+1)\right)\right]$

59. $\frac{d}{dt}\left[\left(t^3+1\right)\cosh\left(t^3+1\right)\right]$

60. $\frac{d}{dz}\left[\frac{1}{z}\tanh\left(z^2\right)\right]$

61. $\frac{d}{dy}\left[\cosh\left(\ln(2y+1)\right)\right]$

62. $\frac{d}{dx}\left[\frac{\sinh(\ln x)}{x+1}\right]$

63–71 Evaluate the given integral.

63. $\int \frac{1+\sinh^2 x}{\cosh x}\,dx$

64. $\int \operatorname{sech}^2\left(\frac{x}{3}+5\right)dx$

65. $\int \frac{\operatorname{sech}\frac{1}{z}\tanh\frac{1}{z}}{z^2}\,dz$

66. $\int \frac{\coth\sqrt{x}}{\sqrt{x}}\,dx$

67. $\int \tanh(2-w)\,dw$

68. $\int x\cosh\left(x^2\right)dx$

69. $\int_0^1 \operatorname{csch}^2(2-t)\,dt$

70. $\int_0^3 \operatorname{sech}^2 u\sqrt{\tanh u}\,du$

71. $\int_0^2 \frac{\sinh t}{\sqrt{4-\cosh^2 t}}\,dt$

72. By differentiating, verify the given identity.

$$\frac{d}{dx}\left[\tan^{-1}(\sinh x)\right] = \frac{d}{dx}\left[\sin^{-1}(\tanh x)\right] = \operatorname{sech} x$$

73–74 Given that the hyperbolic functions can be expressed in terms of exponential functions, it's not surprising that their inverses can be expressed in terms of logarithms. For example, if we let $y = \sinh^{-1} x$, then $x = \sinh y$ and hence

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$e^y - 2x - e^{-y} = 0$$

$$e^{2y} - 2xe^y - 1 = 0 \qquad \text{Multiply through by } e^y.$$

$$\left(e^y\right)^2 - 2xe^y - 1 = 0 \qquad \text{Express as a quadratic in } e^y.$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \qquad \text{Solve for } e^y.$$

$$e^y = x + \sqrt{x^2+1} \qquad x - \sqrt{x^2+1} < 0 \text{ but } e^y > 0, \text{ so discard } x - \sqrt{x^2+1}.$$

$$y = \ln\left(x + \sqrt{x^2+1}\right). \qquad \text{Take the natural logarithm of both sides.}$$

Use the procedure above to verify the identity.

73. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \ge 1$

74. $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1$

(**Hint:** Begin by setting $y = \tanh^{-1} x$; then write the equation as $\tanh y = x$, square both sides, and apply the identity $\tanh^2 y = 1 - \operatorname{sech}^2 y$. Solve the result for $\cosh y$, apply $\cosh^{-1}$ to both sides, and apply the result of the previous exercise. Then apply some logarithmic properties.)

75–77 Given a function f, let $1/f$ denote its reciprocal—that is, $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$. The following is a useful relationship between the functions f^{-1} and $(1/f)^{-1}$, assuming both of these inverse functions exist.

$$\left(\frac{1}{f}\right)\left(f^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{f\left(f^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$$

$$\Rightarrow \left(\frac{1}{f}\right)^{-1}(x) = f^{-1}\left(\frac{1}{x}\right)$$

Applied to inverse hyperbolic functions, this fact indicates the following relationships.

$$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x} \qquad \operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x} \qquad \coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

Use these relationships to verify the equality.

75. $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right); \quad x \neq 0$

76. $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right); \quad 0 < x \leq 1$

77. $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right); \quad |x| > 1$

78–81 Prove the given formula for the derivative of an inverse hyperbolic function.

78. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$

79. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$

80. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}, \quad |x| < 1$

81. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}, \quad |x| > 1$

82–88 Find the given derivative.

82. $\frac{d}{dx}\left[\frac{1}{2}\sinh^{-1}(5x)\right]$

83. $\frac{d}{dx}\left[\cosh^{-1}\left(\frac{7x}{2}\right)\right]$

84. $\frac{d}{dx}\left[\operatorname{sech}^{-1}(\sin x)\right]$

85. $\frac{d}{dx}\left[\tanh^{-1}(\cos 2x)\right]$

86. $\frac{d}{dx}\left[\cosh^{-1}(\sec x)\right]; \quad 0 < x < \frac{\pi}{2}$

87. $\frac{d}{dx}\left[\coth^{-1}\left(\frac{1}{2x}\right)\right]$

88. $\frac{d}{dx}\left(x \cdot 2^{\sinh^{-1} x}\right)$

89–102 Evaluate the given integral.

89. $\int \frac{dx}{\sqrt{16x^2+9}}$

90. $\int \frac{x}{\sqrt{25x^4-4}}\,dx$

91. $\int \frac{dx}{-x^2+6x-8}$

92. $\int \frac{x}{\sqrt{x^4-4x^2+3}}\,dx$

93. $\int \frac{-\cos x}{\sqrt{\sin^2 x + \sin^4 x}}\,dx$

94. $\int_0^{1/8} \frac{2}{1-4x^2}\,dx$

95. $\int_0^5 \frac{10}{\sqrt{1+25x^2}}\,dx$

96. $\int \frac{dx}{\sqrt{9+e^{2x}}}$

97. $\int_{-3/2}^{-1} \frac{-1}{(x+2)\sqrt{-3-4x-x^2}}\,dx$

98. $\int_1^2 \frac{-1}{x\sqrt{1+x^2}}\,dx$

99. $\int \frac{dx}{x\sqrt{1+x^3}}$

100. $\int \frac{dx}{\sqrt{x}\sqrt{1+x}}$

101. $\int_{1/8}^{1/4} \frac{dx}{x\sqrt{1-4x^2}}$

102. $\int_7^9 \frac{-4}{-x^2+6x-5}\,dx$

103–110 Evaluate the given limit.

103. $\lim_{x\to\infty} \frac{\cosh x}{x^2}$

104. $\lim_{x\to 0} \frac{\sinh x}{x}$

105. $\lim_{x\to 0} \frac{\sinh^{-1} x}{x}$

106. $\lim_{x\to 0} \frac{\tanh x}{x}$

107. $\lim_{x\to\infty} \frac{\cosh x}{e^x}$

108. $\lim_{x\to\infty} x\coth^{-1} x$

109. $\lim_{x\to 0} \frac{1-\operatorname{sech} x}{\sinh x}$

110. $\lim_{x\to 0}\left(\operatorname{csch} x - \coth x\right)$

111. To exhibit another nice analogy between trigonometric and hyperbolic functions, prove that the area of both the circular and hyperbolic sectors corresponding to the parameter u in Figures 4 and 5 equals $A = u/2$. (**Hint:** For the circular sector, use the fact that the arc length of the sector is u units, while the radius is 1. For the hyperbolic sector, notice that

$$A(u) = \frac{\cosh u \sinh u}{2} - \int_1^{\cosh u} \sqrt{x^2-1}\,dx,$$

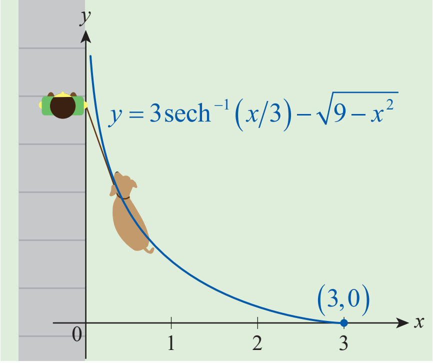
and show by differentiating that $A'(u) = \frac{1}{2}$.)

112. Prove the following interesting property of the hyperbolic cosine function: The area under the graph of $y = \cosh x$ $(0 \leq x \leq c)$ is equal to its arc length over the same interval.

113. A flexible chain or cable suspended between two fixed points forms a curve called a *catenary* (from the Latin word "catenarius," meaning "related to a chain"). The equation of a catenary is $y = a\cosh(x/a)$. Note that the clothesline of Exercise 56 in Section 6.3 is much better approximated by the equation $y = 5\cosh(x/5) - 4$. Use this equation to answer the following questions.

- **a.** What is the slope of the clothesline at each of its endpoints? Compare your answer to the slope predicted by the parabolic model.
- **b.** What is the length of the clothesline? Is your answer close to that given to Exercise 83 in Section 6.3? (Galileo observed that the parabolic model is almost exact when the angle of elevation is less than 45°.)

114.* Suppose that a dog is 3 yards away from a sidewalk, held on a tight leash by its owner, such that the leash is initially perpendicular to the sidewalk. The dog's owner starts walking on the sidewalk at a steady, slow pace, pulling the dog while it is offering slight resistance, thereby keeping the leash tight. The curve of the dog's path in this situation is called a *tractrix* (from the Latin word "trahere," meaning "pull" or "drag"), and can be given by the equation $y = 3\operatorname{sech}^{-1}(x/3) - \sqrt{9 - x^2}$. (We assume that the owner started from the origin, walking along the positive y-axis, while the dog's initial position was $(3,0)$.)



a. Find how far the owner has to walk in order to bring the dog within 1 yd of the sidewalk.

b. Prove that the dog's velocity vector at any time is pointing toward its owner. (For this reason, the tractrix is also called a "curve of pursuit.")

Concept Check

115–118 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

115. $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

116. $|\operatorname{sech} x| \le 1$ for all x

117. The value of $\operatorname{csch} 0$ is infinity.

118. The function $f(x) = \operatorname{sech}^{-1} x$ is defined on $[0,1]$.

6.6 Technology Exercises

119–120 Use a graphing utility to solve the problem.

119. The Gateway Arch of St. Louis (constructed in 1963–1965) was designed by Eero Saarinen so that its central curve (the curve tracing the centroids of the triangular cross-sections) is an inverted, "flattened" catenary that is described by the equation $y = -68.7672\cosh(0.0100333x) + 693.8597$ (units in feet).

a. How tall is the central curve of the arch?

b. How wide is the central curve at ground level?

c. What is the slope of the central curve at an altitude of 600 ft above ground level?

120. Suppose that the clothesline of Exercise 113 became loose, and resembles the parabola $y = \frac{1}{2}x^2 + 0.1$, $-1.5 \le x \le 1.5$ (in particular, it is sagging so that its lowest point is just 0.1 yd above the ground). However, we know that in reality, the clothesline can be much better approximated by a catenary. Find an equation for such a catenary, and graph both curves on the same screen. Do they seem to "overlap," or are they distinguishable? Then graph both the parabolic model and the catenary from Exercise 113 on the same screen. Comparing the two sets of graphs, what can you conclude?

121–122 Use a graphing utility to graph the given function and to find its derivative and any relative extrema.

121. $f(x) = (x^2 - x)\tanh x$

122. $g(x) = x^2(\sinh x - 3\tanh x)$