Multiplying both ends of this last equation by either ρg or δ (depending on the context), we obtain the following result.

$$F = \rho g \bar{y} A = \delta \bar{y} A$$

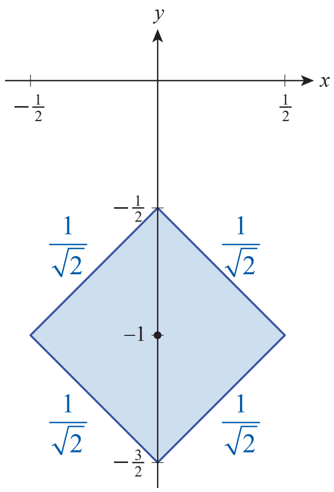


Figure 10

Example 8 Using the Centroid and Area of a Vertical Plate to Find the Fluid Force Exerted on It

Calculate the fluid force on the square cover plate of Example 7 using its centroid and area.

Solution

The centroid of the plate is 1 m from the water's surface, and the area of the square plate is $\frac{1}{2}\text{ m}^2$.

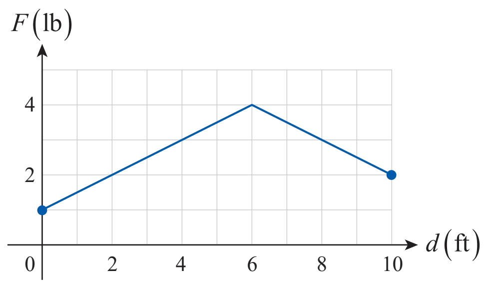
$$\begin{aligned} F = \rho g \bar{y} A &= \left(1000 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\left(1 \text{ m}\right)\left(\frac{1}{2} \text{ m}^2\right) \\ &= 4905 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = 4905 \text{ N} \end{aligned}$$

6.5 Exercises

1. A dad is pushing his child on a sled with a constant, horizontal force a distance of 100 m. The child and sled together have a mass of 35 kg and the coefficient of friction is $\mu = 0.1$. Find the work done against friction. Use $g \approx 9.81 \text{ m/s}^2$. (**Hint:** For a refresher on friction and coefficient of friction, see Example 5 of Section 1.5.)

2. An object of mass 10 kg is pulled 8 m up a 30° ramp. If the coefficient of friction is $\mu = 1/\left(4\sqrt{3}\right)$, find the work done during the process. (**Hint:** Work needs to be done against both friction and gravity. The normal force between the object and the surface of the ramp is $F_{\perp} = mg\cos 30°$.)

3. The graph of a variable force is shown below as it moves an object in a straight line a distance of 10 ft. Find the work done during this process.



4. A 15 cm long unstressed spring requires a force of 3 N to be held stretched to 20 cm. How much work will be done in stretching the spring an additional 5 cm?

5. A force of 50 lb stretches an 8 ft spring by $\frac{1}{2}$ ft. Find the work done in stretching the spring **a.** from its original length to 10 ft and **b.** from 11 ft to 13 ft.

6. A particle is moving along the y-axis from the origin to the point $(0,4)$ under the influence of the variable force $F(y) = \frac{1}{2}y^2 + \sqrt{y}$ (units are in meters and newtons, respectively). Find the work done by the force.

7. If a 3 lb force compresses a 5 ft spring by 18 in., how much work is done in **a.** compressing it from its original length to 3 ft and **b.** stretching it to 6.75 ft?

8. When a mass of 100 g is hung on a vertical spring, the spring is stretched by 4 cm. How much work is done in stretching the spring an additional 5 cm?

9. A bumping post at a railway company has a spring constant of 3.8×10^5 N/cm. Find the work done in compressing this spring by 2 cm.

10. Suppose that $12.5 \text{ ft} \cdot \text{lb}$ of work is needed to stretch a spring from its unstressed length of 1.5 ft to 2 ft. Find the spring constant.

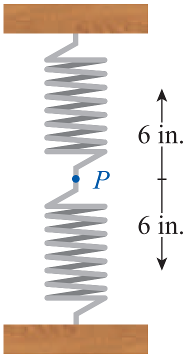
11. If the unstressed length of a spring is 20 cm and 0.12 J of work is required to stretch it from 22 cm to 24 cm, how much work is required to stretch it from 24 cm to 30 cm?

12. Suppose the work done in stretching a spring from 8 in. to 10 in. is $\frac{1}{2}$ ft $\cdot$ lb, and an additional 2 ft $\cdot$ lb of work will stretch it by another 4 in. Use this information to find the unstressed length of the spring.

13. We will call a spring linear if it obeys Hooke's Law (recall that this can only be expected between reasonable limits). Prove that the work done in stretching (or compressing) a linear spring by x units from its original length is $W = \frac{1}{2}kx^2$, where k is the spring constant.

14. Suppose that an elastic rope with an unstressed length of 20 ft behaves in the following, nonlinear manner: the force required to stretch it by x ft is $F = kx^{5/4}$ lb. If a force of 3 lb stretches the rope to 21 ft, how much work is done in stretching it from 21 ft to 22 ft?

15. Two identical springs with a spring constant of 5 lb/ft are attached to each other at point P, with their other ends fastened to the top and bottom of a wooden box, so that they are in a vertical, unstressed position. Find the work done in moving point P vertically up or down by 6 in. (Ignore the weight of the springs.)

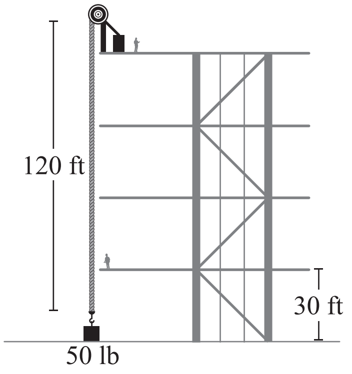


16. A 15 ft chain that weighs 4 lb/ft is hanging from a cylindrical drum so that its other end touches the ground.

a. How much work does it take to wind it up completely?

b. How much work does it take to wind up only two-thirds of the chain?

17. A 120 ft cable is supporting a 50 lb piece of equipment at a construction site. In order to lift the equipment by 30 ft, the cable is wound on a cylindrical drum. If the cable weighs 3 lb/ft, find the work done during this process.



18. A cable starts to unwind from a cylindrical drum at $t = 0$, at a rate of 0.5 m/s. If the cable weighs 20 N/m, find the work done by gravity from $t = 0$ to $t = 12$ s. (**Hint:** Let x denote the length of cable already unwound at time t.)

19. Find the work done in lifting 300 kg of coal from a 400 m deep mine by a rope that weighs 35 N/m. (**Note:** The original term *horsepower* was coined by James Watt after actually watching ponies lift coal from a mine and calculating their work done in unit time.)

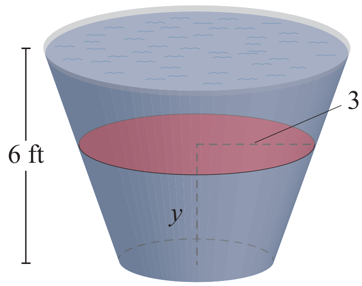
20. A 5 lb bucket is used to draw water for livestock from a 60 ft deep well. The bucket weighs 50 lb when full, but is leaking water at 0.1 lb/s. If it is being pulled at a rate of 1 ft/s, find the work done in getting it to the surface. (Ignore the weight of the rope).

21. A crane is lifting a leaky container full of fresh liquid mortar at a construction site. If the container weighs 40 kg and is able to hold 250 kg of mortar, but is leaking at a rate of 0.6 kg/s and is being lifted at 1.5 m/s for 10 s, find the work done by the crane.

22. A cylindrical tank of depth 5 ft and radius 4.087 ft holds about the same volume as the tank in Example 5. If such a tank is filled with gasoline, determine the work required to pump all the gasoline to the top of the tank, and compare the answer to part a. of Example 4 and to Example 5.

23. Find the work required to pump the gasoline out of the tank of Example 4 over its top if the tank stands on its base. Compare your answer to part a. of Example 4 and explain.

24. A tank in the shape of an inverted cone frustum has cross-sections of radius $3+\frac{1}{3}y$ feet at an altitude of y feet above the base. If its height is 6 feet and it is filled with water, how much work is done to pump all of the water out over the top of the tank? (For the weight density of water, see Example 6.)



25. A rectangular tank of base 5 ft by 8 ft and of height 10 ft is filled with oil. The weight density of oil is $50\ \text{lb}/\text{ft}^3$.

a. Find the work required to pump the oil out of the tank through an outlet on the top.

b. Find the work required if the tank is only half full at the start of pumping.

c. Is the answer you gave in part b. half of that given in part a.? Why?

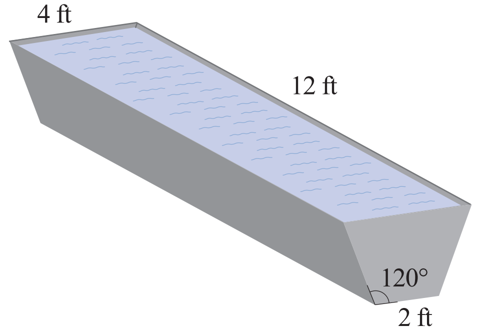
d.* How long will it take for a $\frac{1}{2}$-horsepower pump to empty the tank? What if we start with the tank half full? (One horsepower (hp) is $550\ \text{ft}\cdot\text{lb/s}$.)

26. A cistern in the form of an inverted rectangular pyramid with a square base of side length 2 m and a depth of 4 m is full of water. The weight density of water is $\delta = 9810\ \text{N}/\text{m}^3$.

a. How much work is required to pump all of the water out over the top edge of the cistern?

b. How long will it take for a 1 hp electric pump motor to do the job? One horsepower is 746 watts (W), where $1\ \text{W} = 1\ \text{J/s}$.

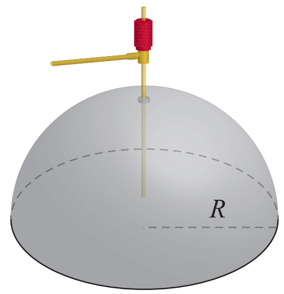
27. The two ends of a watering trough are isosceles trapezoids sitting on the shorter base which is 2 ft, with the legs making 120° angles with that base. The opening of the trough is a 4 ft by 12 ft rectangle. Starting with a full trough, find the energy expended (i.e., total work done) in pumping all of the water out of the trough over its top edge. (For the weight density of water, see Example 6.)



28. Answer Exercise 27 if the ends of the trough are semicircular with a radius of 2 ft.

29. A water trough with the same top opening as that in Exercise 27 has vertical cross-sections in the form of an equilateral triangle. Find the work required to fill it with water through an opening on its bottom.

30. How much work is necessary to pump all fluid of weight density δ out of a hemispherical container of radius R through an opening in its top?



31. Find the work required if the tank in Exercise 30 is inverted.

32. The shape of a kerosene tank can be approximated by rotating the graph of $y = x^4$, $-2 \le x \le 2$, about the y-axis (units in meters). Find the work required to fill up the empty tank through an opening at its lowest point $x = 0$. The weight density of kerosene is $8016.24\ \text{N}/\text{m}^3$.

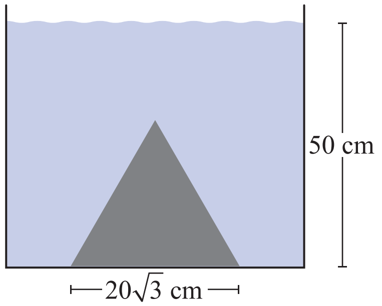
33. According to Archimedes' Principle, the buoyant force acting on a body immersed in fluid is equal to the weight of the fluid it displaces. Use this principle to find the work required to completely immerse in water a cube of negligible weight if its edges are 1 ft. (**Hint:** Keep the top horizontal during immersion.)

34. Repeat Exercise 33 for a buoy that is a circular cone of base radius R and height h, when it is immersed vertically, vertex first in a fluid of weight density δ.

35–43 Use the integral formula for fluid force to answer the question.

35. A 2 ft by 3 ft rectangular plate is positioned vertically on the bottom of an 8 ft pool. Find the fluid force exerted on one side of the plate if it sits **a.** on its long edge and **b.** on its short edge.

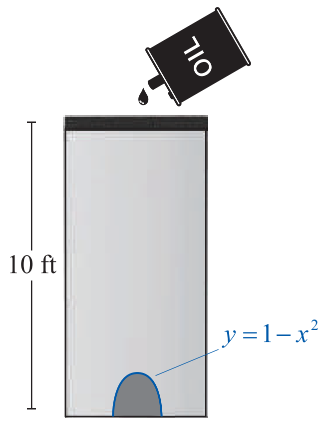
36. An equilateral triangle of side length $20\sqrt{3}$ cm is standing on its base that is 50 cm underwater. Find the fluid force against one face of the triangle. Use $\delta = 9.81 \cdot 10^{-3}\ \text{N/cm}^3$ for the weight density of water.



37. Find the force against one face of the triangle from Exercise 36 if the upper third of its vertical altitude is sticking out of the water.

38.* Answer Exercise 36 if the triangle is tilted, making a 30° angle with the horizontal.

39. Suppose that a vertical, parabolic gate is installed on a vertical side of a 10 ft deep gas tank. The gate is given by the equation $y = 1 - x^2$, so that $y = 0$ coincides with the bottom edge of the tank (units in feet). If the gate is designed to withstand a maximum force of 600 lb, verify that it won't break when gas is stored in the tank. Will the gate break if the tank is filled up with oil? The weight densities of gas and oil are $44\ \text{lb/ft}^3$ and $50\ \text{lb/ft}^3$, respectively.

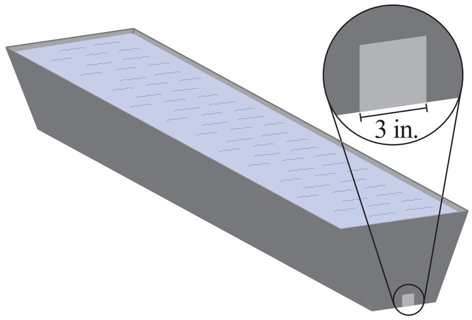


40. Find the force exerted on the end of the full trough in Exercise 27.

41. Find the force exerted on the end of the full trough in Exercise 29.

42.* Find the force exerted on one end of a trough with the same ends and opening as the one in Exercise 29, but with its ends tilted outward by 45°.

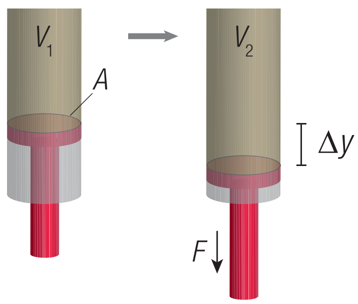
43. Suppose that an outlet near the bottom of the trough in Exercise 27 is covered by a 3 in. by 3 in. square plate, so that the lower edge of the plate coincides with that of the trough. What force is pressing against the plate?



44–47 Consider a certain gas of pressure P and volume V_1 confined in a cylinder, closed on one end by a moveable piston. If A is the area of the piston, the force acting on the piston is $F = PA$. Thus, as the gas expands, pushing the piston by a small increment of Δy, the work done on the piston is $\Delta W = F\Delta y = PA\Delta y = P\Delta V$. Integrating, we obtain the work done by the gas as it expands from a volume of V_1 to V_2.

$$W = \int_{V_1}^{V_2} P\,dV$$

(**Note:** This argument can be "reversed" to obtain the work required to compress the gas.)



In Exercises 44–47, apply the above formula.

44. Assume that 0.5 ft³ of gas in a cylinder under an initial pressure of $300\ \text{lb/ft}^2$ expands to 1.5 ft³. Assuming that $PV = c$ (a constant) during this process, find the work done by the gas on the piston.

45. Suppose that 3 L of gas under initial pressure of 150 kPa in a cylinder is compressed to 1 L. Find the work done by the piston. (Assume, as in Exercise 44 that $PV = c$.)

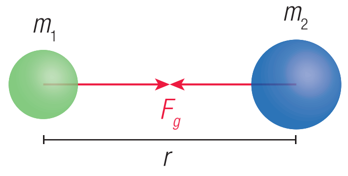
46. Repeat Exercise 44 under the assumption that $PV^{1.4} = c$. (This happens when heat loss is negligible. We say that in this case, P and V are related adiabatically.)

47. Repeat Exercise 45 under the assumption that $PV^{1.4} = c$.

48–55 According to Newton's Law of Gravitation, two masses m_1 and m_2 attract each other by a force that is directly proportional to the product of their masses and inversely proportional to the square of their distance (or rather the square of the distance between their respective centers of gravity):

$$F_g = \frac{m_1 m_2 G}{r^2},$$

where G is the universal gravitational constant. Its value in metric units is $6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.



48. Use the above formula to show that the acceleration caused by gravity on a free-falling mass near Earth's surface is approximately

$$g = \frac{MG}{R^2},$$

where M and R are the mass and radius of Earth, respectively (we are assuming that Earth is perfectly spherical).

49. Show that if an object of weight w is launched to a height h above the surface of Earth, then Earth attracts it by a force of

$$F_g(h) = \frac{R^2 w}{(R+h)^2}.$$

(**Hint:** Use the fact that $w = mg$ along with the result of Exercise 48.)

50. Find the work done against gravity in moving a 3-pound object to an altitude of 500 miles above the surface of Earth. Assume that the radius of Earth is approximately 4000 miles. (**Hint:** Integrate the variable force obtained in Exercise 49 between appropriate limits.)

51. Find the work done against gravity in moving a 2-ton satellite to an altitude of 200 miles above the surface of Earth. Express your answer in the units foot-pounds. (**Hint:** Ignore the work done to accelerate the spacecraft, air resistance, as well as the weight of the launching vehicle and fuel.)

52. How much energy is expended (i.e., work done) in lifting a rocket of mass 10 metric tons to an altitude of 300 km above Earth? Assume the radius of Earth is approximately 6371 km. (See the hint given in Exercise 51.)

53. Calculate the work done in Exercise 52 if the rocket is launched from the moon. The acceleration due to gravity on the moon is $1.6\ \mathrm{m/s^2}$, about $\frac{1}{6}$ of that on Earth. The moon's radius is approximately 1737 km.

54. Find the work done in moving a spacecraft of mass 110 metric tons to an altitude of 300 miles above the surface of Earth. Use 1 mile ≈ 1600 m and $g \approx 9.81\ \mathrm{m/s^2}$. Express your answer in megajoules. (**Hint:** Ignore the work done to accelerate the spacecraft, air resistance, as well as the weight of the launching vehicle and fuel. See Exercise 52 for Earth's radius.)

55. Find the limit of the work done in moving a weight w to a distance d above Earth's surface as $d \to \infty$.

56–57 The magnitude of the force acting between two point charges q_1 and q_2 at a distance of r units from each other, is described by Coulomb's Law, as follows.

$$F = k\frac{|q_1 q_2|}{r^2}$$

Notice the analogy between Coulomb's Law and the law of gravity! The charges are analogous to the masses, while k is analogous to the universal gravitational constant. The approximate value of k is $8.98755 \times 10^9\ \mathrm{Nm^2/C^2}$. (A coulomb (C) is the SI unit for electric charge. Note that all units are SI; there is no British system of electrical units.) It is also worth noting that Coulomb's Law only gives the magnitude of the force, since electric forces can be attractive or repulsive, while the force of gravity is always attractive.

In Exercises 56–57, use Coulomb's Law.

56. Two like electrical charges of 10^{-4} C each, are 50 cm apart, with one of them fixed. Find the work done in bringing the other charge to a distance of 20 cm from the fixed charge.

57. If two point charges of opposite sign attract each other by a force of 200 N when 3 cm apart, find the work done in moving them from 2 cm apart to 8 cm apart.

58–64 The centroid can be used in some circumstances to simplify work calculations, as well as fluid force calculations. For instance, when the 5 lb rope in Example 3 is extended 50 ft down the well, it behaves, from a center of mass perspective, like a 5 lb point mass 25 ft down the well. The work required to lift such a point mass to the surface is $(5 \text{ lb})(25 \text{ ft}) = 125 \text{ ft} \cdot \text{lb}$. Added to the $500 \text{ ft} \cdot \text{lb}$ of work required to lift the 10 lb bucket 50 ft, we get the total work required: $625 \text{ ft} \cdot \text{lb}$.

In Exercises 58–64, use this centroid argument to find a second solution.

58. Exercise 16

59. Exercise 17

60. Exercise 18

61. Exercise 19

62. Exercise 25

63. Exercise 28 (**Hint:** Use Exercise 52 of Section 6.4.)

64. Exercise 29

65–68 Use the technique of Example 8 to find a second solution.

65. Exercise 33

66. Exercise 34

67. Exercise 36

68. Exercise 41

69. Prove the following statement: If two cylindrical tanks hold the same volume, but the height of tank B is half of the height of tank A, then the work required to pump all liquid out of a full tank A over its top is twice the work required to do the same for tank B. Do the proof in two ways: **a.** by using integration and **b.** by using the centroid approach.

6.5 Technology Exercises

70–71 Use a graphing utility to solve the problem.

70. Suppose that the rocket in Exercise 52 runs out of fuel after 1.9×10^7 kJ of useful energy is expended. How high will the rocket go?

71. Suppose that in Exercise 25 part d., the pump stops due to an electrical failure 3 minutes after starting the job. What is the fluid level at that instant?