6.4 Exercises

1–3 Find the moment M_0 about the origin and the center of mass $\bar{x}$ for the point masses located on the x-axis.

1. $m_1 = 4, x_1 = -3; \quad m_2 = 5, x_2 = -1; \quad m_3 = 2, x_3 = 6$

2. $m_1 = 2, x_1 = -5; \quad m_2 = 10, x_2 = -2;$
$m_3 = 8, x_3 = 1; \quad m_4 = 3, x_4 = 7$

3. $m_1 = 3.5, x_1 = -10; \quad m_2 = 5, x_2 = -2;$
$m_3 = 2, x_3 = 5.5; \quad m_4 = 2.5, x_4 = 11$

4–6 Find the moments M_x, M_y about the coordinate axes and the center of mass for the system of point masses.

4. $m_1 = 4, P_1(-6,-8); \quad m_2 = 5, P_2(1.5,2);$
$m_3 = 2, P_3(3,4)$
(What do you notice about $\bar{x}$?)

5. $m_1 = 2, P_1(-2,6); \quad m_2 = 4, P_2(-1,-5);$
$m_3 = 7, P_3(2,0); \quad m_4 = 8, P_4(3,4)$

6. $m_1 = 2.5, P_1(-9,0); \quad m_2 = 3, P_2(-4.5,0.25);$
$m_3 = 2, P_3(0,-2.5); \quad m_4 = 6, P_4(2.25,5);$
$m_5 = 1.5, P_5(5.5,-3.5)$

7. Tyler and Christina are sitting on the ends of a 14-foot seesaw, unable to balance it because of their weight difference. However, Tyler's little sister, Lisa, is quick to come to the rescue. If Tyler and Christina weigh 110 and 80 pounds, respectively, while little Lisa is 35 pounds, where should she sit in order for the seesaw to balance? (**Note:** Strictly speaking, when you are multiplying weight by distance, you are calculating torque rather than moment, but the technique in obtaining balance is the same.)

8. The design of a certain front-wheel-drive family sedan allows for 54% of its total mass to rest on the front axle, while 46% is resting on the rear axle. The distance between the front and rear axles (the wheelbase of the car) is 2.65 meters. How far behind the front axle is the car's center of mass?

9. An experimental rocket is fired toward the north and is on track to hit the target when a midair explosion breaks it apart. The tail section, which is twice as heavy as the nosepiece, is found 200 yards southwest from the intended target. Where is the nosepiece likely to be found? (**Hint:** Even after breaking apart, the center of mass of the rocket will arrive in the target area.)

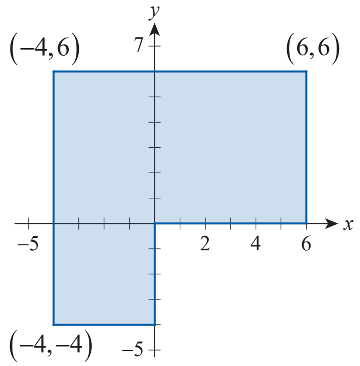
10–11 A thin rod of length l, with the given continuously varying density is placed into the coordinate system so that it lies horizontally on the x-axis with its left endpoint coinciding with the origin. Find its center of mass.

10. $l = 2.25 \text{ m}, \quad \rho = 1 + \sqrt{x} \text{ kg/m}$

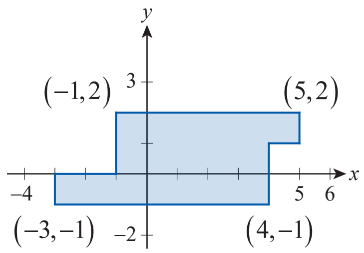
11. $l = 110 \text{ cm}, \quad \rho = 0.6 + 0.01x \text{ g/cm}$

12–15 Use the indicated coordinates to determine the center of mass of the given region. (**Hint:** Divide the region into appropriate subregions and treat the centers of mass of the subregions as point masses.)

12.



13.



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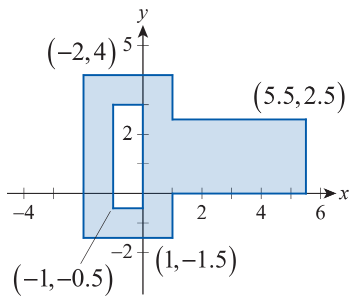
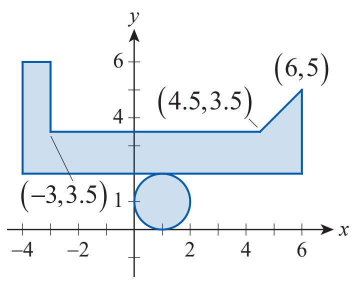


15.



16. Show that if the plate of Example 3 has constant density, then its center of mass is located at $\left(\frac{3}{4}, \frac{3}{10}\right)$.

17–40 Find the centroid of the plane region bounded by the given curves. If possible, use symmetry to simplify your calculations. (In Exercise 31, use the formula $\cos^2 x = (1+\cos 2x)/2$ before integrating.)

17. $y = x^2 + 4x, \quad y = 0$

18. $y = \sqrt{4-x}, \quad x = 0, \quad y = 0$

19. $y = 2x^2, \quad 32x + y^2 = 0$

20. $y = \dfrac{x^2}{4}, \quad x = 2, \quad y = 0$

21. $y = \dfrac{\sqrt{x}}{2} + 1, \quad y = \dfrac{1}{4}x + 1$

22. $y = x^2, \quad y = x^3$

23. $y = 3 - x, \quad x = 0, \quad y = 0$

24. $x + 2y = 13, \quad 2x - y + 4 = 0, \quad x = 0$

25. $y = \dfrac{1}{x}, \quad x = 1, \quad x = 3, \quad y = 0$

26. $y = \sqrt{2x}, \quad y = x$

27. $y = x^2 - 9, \quad y = 0$

28. $y = x^2 - 9, \quad x \geq 0, \quad y = 0$

29. $y = x^3, \quad y = 4x, \quad x \geq 0$

30. $y = \sqrt{x}, \quad y = x^2$

31.* $y = \cos^2 x, \quad y = 0, \quad 0 \leq x \leq \pi$

32. $xy = 1 \quad x = 1, \quad y = 3$

33. $y = \dfrac{1}{x^2}, \quad y = -\dfrac{1}{x^2}, \quad 1 \leq x \leq 2$

34. $y = x^{2/3}, \quad x = 0, \quad y = 1$

35. $y = (x+2)^2, \quad y = (x-2)^2, \quad y = 0$

36. $y = 2\sqrt{x+4}, \quad 2y = 8 - x, \quad y = 0$

37. $y = x + 1, \quad y(x+1) = 1, \quad y = 0, \quad x = 1$

38. $y = \sqrt{3-x}, \quad y = 2\sqrt{-x}, \quad y = 0$

39. $4x^2 + 9y^2 = 49, \quad x \geq 0, \quad y \geq 0$

40. $x^4(1-x^2) = y^2, \quad x \geq 0, \quad y \geq 0$

41–50 Find the center of mass of the plane region of varying density that is bounded by the given curves.

41. $3y = 6 - x, \quad x = 0, \quad y = 0; \quad \rho(x,y) = x$

42. $3y = 6 - x, \quad x = 0, \quad y = 0; \quad \rho(x,y) = 1 + y$

43. $y = x^{3/2}, \quad y = \sqrt{x}; \quad \rho(x,y) = \sqrt{x}$

44. $y = \sqrt{1-x^2}, \quad x \geq 0, \quad y = 0; \quad \rho(x,y) = x$

45. $y = 4 - x^2, \quad y = 2 - x; \quad \rho(x,y) = 2 + x^2$

46. $y = \dfrac{3}{x^3}, \quad x = \dfrac{1}{2}, \quad x = 1, \quad y = 0; \quad \rho(x,y) = 1 - x^3$

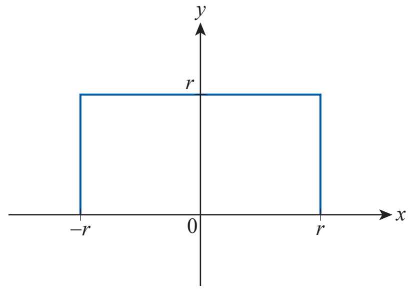
47. $y = \sqrt[4]{x}, \quad y = \sqrt{x}; \quad \rho(x,y) = 1 - y$

48. $xy^2 = 3, \quad x = 0, \quad y = \dfrac{1}{3}, \quad y = 1; \quad \rho(x,y) = y^3 - y^2$

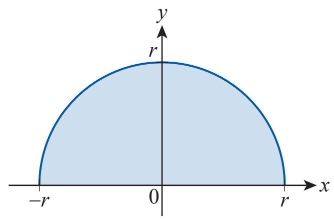
49. $y + 2x = 5, \quad x = y + 1, \quad y = 0; \quad \rho(x,y) = 5 - x$

50. $x = \sqrt{y}, \quad xy = 1, \quad x = 0, \quad y = 2; \quad \rho(x,y) = y$

51. Find the center of mass of the wire if we modify Example 4 by bending the wire to form three sides of a rectangle as shown in the figure below.



52. Find the centroid of the half disk shown in the figure below. Assume constant density.



53. Find the centroid of the first quadrant of the half disk in Exercise 52.

54.* Find the center of mass of the half disk of Exercise 52 if its density function is $\rho(x,y) = y$. (**Hint:** To eliminate the radical, use the substitution $x = r\sin\theta$ where applicable.)

55. Find the center of mass of the wire in Example 4 if it has a variable density of $\rho(\theta) = |\cos\theta|$.

56.* Prove that the center of mass of a triangle is the intersection of its medians. (**Hint:** Recall what you learned in geometry about the intersection point of the medians.)

57. Prove that the distance of the center of mass of a triangle from each side is one third of the corresponding altitude. (**Hint:** Try to prove the result for right triangles first.)

58.* The graph of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse centered at the origin. Find the centroid of the plane region bounded by the first quadrant of this ellipse and the coordinate axes.

59. Find a formula for the centroid of the region bounded by $y=\sqrt[n]{x}$, $y=0$, and $x=1$ as a function of n. What can you say if $n\to\infty$?

60. Use Pappus' Theorem for volumes to answer Exercise 52 provided that you know the formula for the volume of the sphere: $V=\frac{4}{3}\pi r^3$.

61. Use Pappus' Theorem for volumes to find the centroid of a right triangle. (**Hint:** Place a right triangle into the coordinate system so that its vertices coincide with the origin and the points $(h,0)$ and $(0,R)$. Then rotate the triangle and use the volume formula for the right circular cone, along with Pappus' Theorem.)

62. Use Pappus' Theorem to find the volume of the solid resulting from revolving the triangle with vertices $(-3,1)$, $(4,1)$, and $(2,7)$ about the line $y=-2$.

63. Rotate the half disk of Exercise 52 about the line $y=r$. Use Pappus' Theorem to find the volume of the resulting solid.

64. Rotate the region of Exercise 24 about the line $y=-4$. Use Pappus' Theorem to find the volume of the resulting solid.

65. Rotate the region of Exercise 28 about the line $y=1$. Use Pappus' Theorem to find the volume of the resulting solid.

66. Rotate the region of Exercise 30 about the line $y=3$. Use Pappus' Theorem to find the volume of the resulting solid.

67. Use Pappus' Theorem to find the volume of the solid generated by revolving the rectangle with vertices $(-1,4)$, $(5,4)$, $(5,8)$, and $(-1,8)$ about the line $y=-x-2$.

68. Use the result of Exercise 40 along with Pappus' Theorem to find the volume of the solid obtained by revolving the right loop of the curve $x^4\left(1-x^2\right)-y^2=0$ about the y-axis.

69. Use the result of Example 4 along with Pappus' Theorem to find the surface area of the solid in Exercise 63.

70. Use Pappus' Theorem for surface areas to verify that the lateral surface area of a right circular cone of slant height s and base radius r is $A=\pi rs$. (**Hint:** Appropriately place a line segment of length s in the coordinate system and rotate to obtain the cone.)

71. Generalize your solution to Exercise 70 to verify the formula for the lateral surface area of a frustum of a cone (for the definition, see the discussion preceding Example 3 in Section 6.3).

Concept Check

72–75 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

72. The center of mass of a thin triangular plate always coincides with the intersection of its medians (this point is also called the centroid of the triangle).

73. The center of mass of an object is always a point on the object itself.

74. Pappus' Theorem makes integration unnecessary for calculating volumes.

75. If a child is 50% heavier than another child, then in order for them to balance on a seesaw, he or she has to sit 50% closer to the pivot point.