Example 8 Using Integration in Multiple Ways to Derive the Formula for Area of a Circle

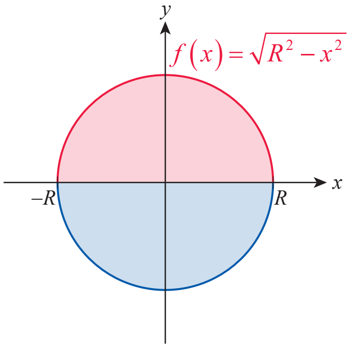


Figure 19

You know that the area of a circle of radius R is $A=\pi R^2$, and one straightforward way to arrive at this formula is to integrate the function $f(x)=\sqrt{R^2-x^2}$ over the interval $[-R,R]$ and then double the result (see Figure 19). However, this is not the easiest way (finding an antiderivative of f is best done using a technique we will learn in Chapter 7). Here are two other approaches.

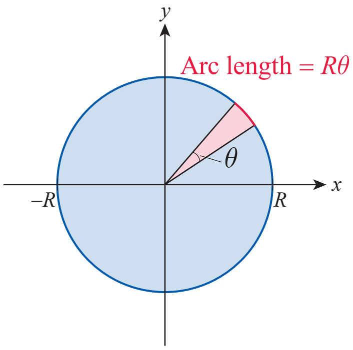


Figure 20

1. The area of a triangle is half the product of its base and height. A thin sector of our circle is approximately a triangle (and the approximation gets better and better as the angle approaches 0), with the base of the triangle equal to $R\,d\theta$ and the height equal to R (see Figure 20).

$$dA=\left(\frac{1}{2}\right)(R)(R\,d\theta)=\frac{1}{2}R^2\,d\theta$$

$$A=\int dA=\int_0^{2\pi}\frac{1}{2}R^2\,d\theta=\frac{1}{2}R^2\,\theta\Big]_{\theta=0}^{\theta=2\pi}=\pi R^2$$

2. If we assume the formula for a circle's circumference is known, then we can decompose our given circle into a sequence of thin concentric rings with each one having area approximately equal to its circumference times dr (see Figure 21).

$$dA=2\pi r\,dr$$

$$A=\int dA=\int_0^R 2\pi r\,dr=2\pi\cdot\frac{r^2}{2}\Big]_{r=0}^{r=R}=\pi R^2$$

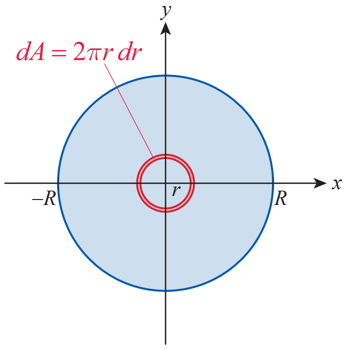


Figure 21

Example 9 Using Integration to Derive the Formula for Volume of a Sphere

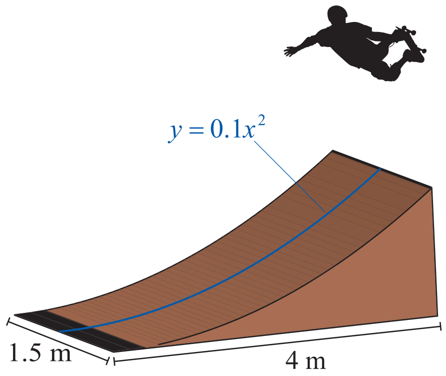
Assume the formula for the surface area of a sphere of radius r is known: $A=4\pi r^2$ (we will soon find this formula on our own). Then an alternative way of finding the volume of a sphere of radius R is to decompose it into a sequence of thin concentric shells, each with volume approximately equal to its surface area times dr.

$$dV=4\pi r^2\,dr$$

$$V=\int dV=\int_0^R 4\pi r^2\,dr=4\pi\cdot\frac{r^3}{3}\Big]_{r=0}^{r=R}=\frac{4}{3}\pi R^3$$

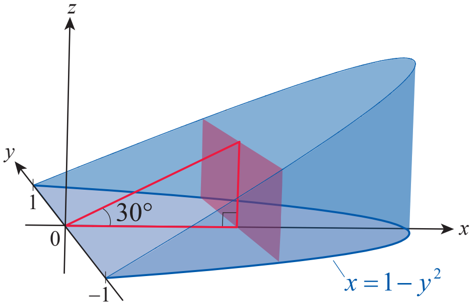
6.1 Exercises

1. Find the volume of a skateboard ramp with a rectangular base of 4 meters by 1.5 meters, if each vertical cross-section is congruent to the "parabolic triangle" bounded by the x-axis, the vertical line $x=4$, and the graph of $y=0.1x^2$ (longitudinal units are meters).

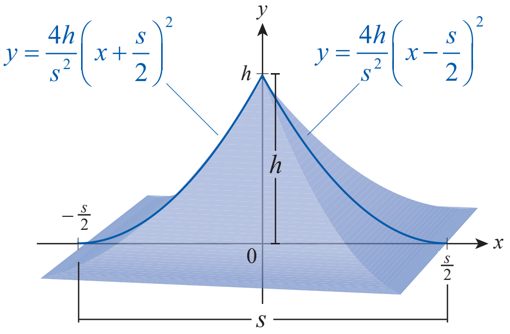


2. Mimic Example 2 to derive the formula for the volume of the right circular cone of height h, if the radius of its base is r.

3. Let's modify Example 3 by assuming that the cylinder is parabolic, that is, the base of the wedge is bounded by the graph of $x = 1 - y^2$ and the y-axis. Moreover, suppose that the second plane makes a 30° angle with the xy-plane. Find the volume of the curved wedge under these conditions.

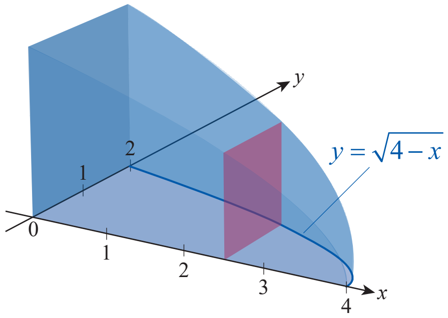


4. Suppose we "carve out" the faces of the pyramid of Example 2 so that the "side view" (i.e., the perpendicular cross-section that contains the altitude and is parallel to a pair of base edges) becomes the region bounded by the graphs of $y = \frac{4h}{s^2}\left(x - \frac{s}{2}\right)^2$, $y = \frac{4h}{s^2}\left(x + \frac{s}{2}\right)^2$, and the x-axis $(-s/2 \le x \le s/2)$. Find the volume of the resulting "concave pyramid."

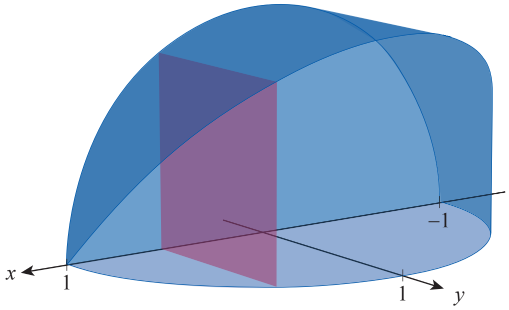


5–14 The base of a solid S is described in the xy-plane along with its cross-sections in a certain direction. Find the volume of S.

5. The base of S is the region bounded by the graph of $y = \sqrt{4 - x}$ and the coordinate axes. The cross-sections perpendicular to the x-axis are squares.



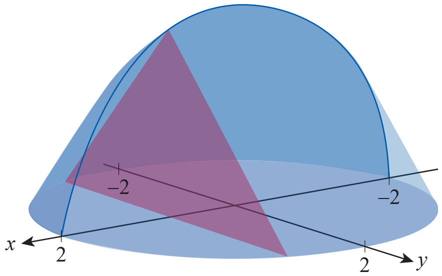
6. The base of S is a half disk of radius 1, and the cross-sections perpendicular to the diameter are squares.



7. Repeat Exercise 6 if the cross-sections are equilateral triangles.

8. The base of S is a disk of radius 2, and the cross-sections perpendicular to the base are rectangles, each with a height that equals twice the width.

9. The base of S is a disk of radius 2, and the cross-sections perpendicular to the base are isosceles triangles, each with a height half as long as its base.



10. The base of S is bounded by $y = x^3$, the y-axis, and the line $y = 8$. Cross-sections perpendicular to the y-axis are squares.

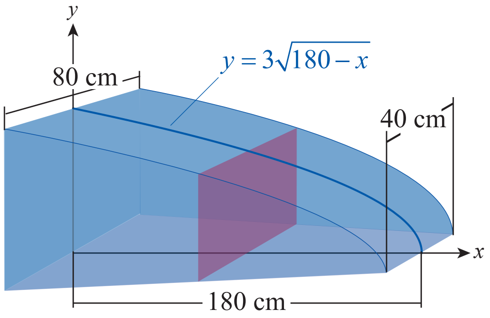
11. The base of S is the region bounded by $y = x^4$ and $y = x^2$, $0 \le x \le 1$, and each cross-section perpendicular to the x-axis is an isosceles right triangle with the right angle's vertex sitting on $y = x^2$.

12. The base of S is bounded by the x-axis and $y = \sin x$, $0 \le x \le \pi$. Each of its cross-sections perpendicular to the x-axis is an isosceles triangle of altitude 1.

13. The base of S is bounded by $y = \cos x$ and $y = \left(\frac{2}{\pi}x\right)^2 - 1$, $-\pi/2 \le x \le \pi/2$. Each of its cross-sections perpendicular to the x-axis is an isosceles triangle of altitude 2.

14. The base of S is bounded by $y = \sqrt{4-x}$, $y = 0$, and $x = 0$. Each of its cross-sections perpendicular to the x-axis is a rectangle of perimeter 10.

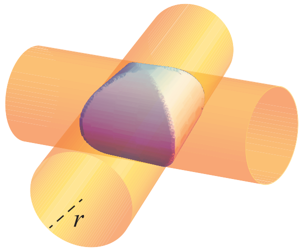
15.* A solid modeling the nose of a race car has a base in the shape of an isosceles trapezoid with a height of 180 cm and base lengths of 80 cm and 40 cm, respectively. Suppose the vertical cross-section through the axis of symmetry of the base (i.e., the cross-section that "cuts the model in half" longitudinally) is the region bounded by the graph of $y = 3\sqrt{180-x}$ and the coordinate axes. All vertical cross-sections perpendicular to the axis of symmetry of the base are rectangles. Find the volume of this model.



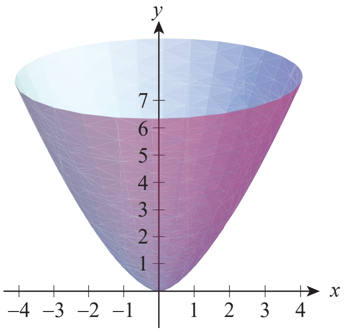
16. Find the volume of water remaining in a spherical reservoir of radius r if the water's depth is $r/3$.

17. A piece in a wooden toy set is a sphere of radius 2 cm, with a cylindrical hole of radius 1 cm drilled through the center. Find the volume of this piece.

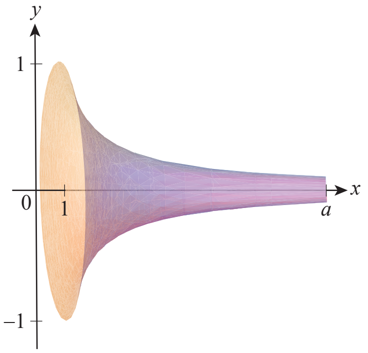
18.* Two plastic pipes (with circular cross-sections) of radius r inches cross at right angles. Find the volume of the solid region that is common to both pipes. (**Hint:** Find cross-sections that are squares.)



19. Find the volume of the "bowl" that results from rotating the graph of $y = x^{3/2}$, $-4 \le x \le 4$, around the y-axis.



20. Find the volume of the solid that results from rotating the graph of $y = 1/x$, $1 \le x \le a$, around the x-axis (this solid is known as *Gabriel's horn*). What can you say if $a \to \infty$?



21–28 Find the volume of the solid that results from rotating the region between the graph of the given equation and the x-axis about the x-axis over the indicated interval.

21. $y = -\frac{1}{3}x + 3; \quad 0 \le x \le 3$

22. $y = 2x + \frac{2}{3}; \quad 0 \le x \le 1$

23. $y = x^2 + \frac{1}{2}; \quad 0 \le x \le 2$

24. $y = x^3; \quad -1 \le x \le 3$

25. $y = -\sqrt{5-x}; \quad 1 \le x \le 4$

26. $y = x^{5/3}; \quad -1 \le x \le 1$

27. $y = \sqrt{9-x^2}; \quad 0 \le x \le 3$

28. $y = \sec x; \quad 0 \le x \le \pi/4$

29–36 Find the volume of the solid that results from rotating the region bounded by the graphs of the equations about the y-axis.

29. $y = \frac{1}{2}x - 1, \quad x = 0, \quad y = 0$

30. $2y + 6x = 9, \quad x = 0, \quad y = 0$

31. $y = (x-1)^2, \quad x = 1, \quad x = 4, \quad y = 0$

32. $y = x^3, \quad x = 2, \quad y = 0$

33. $y = \sqrt[3]{x} + 1, \quad x = 0, \quad y = 3$

34. $y = x^{3/5} - 2, \quad x = 0, \quad y = -2, \quad y = 6$

35. $y = \sqrt{4 - x^2}, \quad x = 0, \quad y = 0$

36.* $y = \csc^{-1} x, \quad x = 0, \quad y = \pi/4, \quad y = \pi/2$

37–48 Find the volume of the solid generated by rotating the region bounded by the graphs of the given equations about **a.** the x-axis and **b.** the line $y = -1$.

37. $y = \frac{1}{2}x, \quad y = 0, \quad x = 2$

38. $y - 3x - 1 = 0, \quad y = 0, \quad x = 1$

39. $y = x^3, \quad y = 0, \quad x = 1$

40. $y = 2\sqrt{x}, \quad y = 0, \quad x = 4$

41. $y = \frac{x^4}{4}, \quad y = x^2, \quad x = 0, \quad x = 2$

42. $y = \frac{2}{x+1}, \quad y = 0, \quad x = 0, \quad x = 2$

43. $y = x^{5/3}, \quad y = 0, \quad x = 0, \quad x = 1$

44. $y = e^x, \quad y = 0, \quad x = 0, \quad x = 1$

45. $y = \sqrt[3]{x}, \quad y = 0, \quad x = 1, \quad x = 8$

46. $y = \frac{1}{2}x^{2/3}, \quad y = \frac{1}{2}x^{3/2}, \quad x = 0, \quad x = 1$

47. $y = \sqrt{1 - (x-1)^2}, \quad y = 0, \quad x = 2$ (**Hint:** When evaluating $\int_0^2 \sqrt{1-(x-1)^2}\, dx$, use a well-known formula from geometry.)

48. $y = \cos x, \quad y = 0, \quad x = 0, \quad x = \pi/2$
(**Hint:** When evaluating the integral, use the identity $\cos^2 x = (1 + \cos 2x)/2$ or the identity $\cos^2 x + \sin^2 x = 1$.)

49–57. In each of Exercises 39–47, rotate the region about **a.** the y-axis and **b.** the line $x = -2$, and find the volume of the resulting solid. (For Exercise 54, use the antiderivative $\int (\ln x)^2\, dx = 2x - 2x \ln x + x(\ln x)^2$. For Exercise 57, see the hint given in Exercise 47.)

58–71 Find the volume of the solid that results from rotating the region bounded by the graphs of the equations about the indicated line. Use any of the methods discussed in this section.

58. $y = -2x + 5, \quad y = -1, \quad x = 0;$ about $x = -1$

59. $y - 4x - 3 = 0, \quad y = 0, \quad x = 0, \quad x = 3;$ about $x = -3$

60. $y = x^2 - \sqrt{8x}, \quad y = 0, \quad x = 0, \quad x = 2;$
about the x-axis

61. $x = 2y(y-5), \quad x = 0;$ about the y-axis

62. $x\sqrt{1+y^2} = 1, \quad x = 0, \quad y = 0, \quad y = \sqrt{3};$
about the y-axis

63. $x = y^2, \quad y = x^2;$ about $y = 1$

64. $y = \sqrt[4]{2x+1}, \quad y = 0, \quad x = 4;$ about the x-axis

65. $y = (x-2)\sqrt{x^2+1}, \quad y = 0, \quad 0 \le x \le 2;$
about the x-axis

66. $y = \sqrt{-x\cos(x^2 + \pi)}, \quad y = 0, \quad x = 0, \quad x = \sqrt{\pi/2};$
about the x-axis

67. $x = \sqrt{y} + 2, \quad 4x = y^2, \quad y = 0, \quad y = 4;$
about the y-axis

68. $y = \sin^{-1} x, \quad y = 0, \quad x = 1;$ about the y-axis (See the hint given in Exercise 48.)

69. $y = \cos x, \quad y = \sin x, \quad x = 0, \quad x = \pi/4, \quad y = 0;$
about the x-axis

70. $x = e^y + e^{-y}, \quad x = 0, \quad y = 0, \quad y = 1;$
about $x = -e$

71. $y = \csc x, \quad y = \cot x, \quad x = \pi/4, \quad x = \pi/2;$
about the x-axis

72–79 The given integral represents the volume of a solid of revolution. Describe the solid. (Do not evaluate the integral.)

72. $\int_0^{\sqrt{3}} 3\pi y^2\, dy$

73. $\int_0^4 \pi x\, dx$

74. $\int_0^2 \pi(4 - y^2)\, dy$

75. $\int_0^1 (\pi e^2 - \pi e^{2x})\, dx$

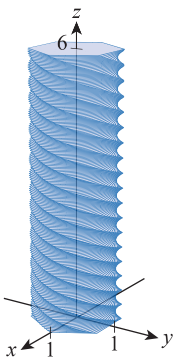
76. $\int_0^1 \pi(\arctan y)^2\, dy$

77. $\pi\int_0^1 (\sqrt{x} - x^8)\, dx$

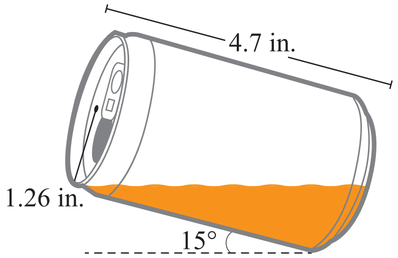
78. $\int_0^{\pi} \pi \sin^2 x\, dx$

79. $\pi\int_1^2 \left[(\log_2 x)^2 - (x-1)^2\right] dx$

80. A regular hexagon of side length 1 lies in the xy-plane so that its center of symmetry coincides with the origin. Suppose the hexagon moves 6 units vertically upward so that its center rides on the line perpendicular to the xy-plane (the line is actually the z-axis), the hexagon at any time is parallel to the xy-plane and makes three revolutions around its center (at a constant angular speed) as it moves. Find the volume of the resulting corkscrew-type solid.

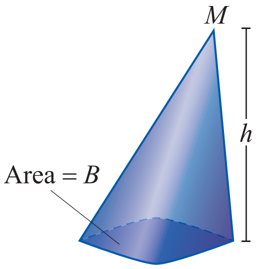


81.* A student is drinking from a standard soda can. Find the volume of the remaining soda when the can's axis of symmetry makes a 15° angle with the horizontal direction. Express your answer in milliliters. (**Hint:** Approximate the soda can with a right circular cylinder of radius 1.26 inches and height of 4.7 inches. Assume that when the can is held as described in the problem, the soda is level with the lowest point of the top rim of the can.)



82.* Let D denote the depth of water in a bowl that has a tiny hole in the bottom. According to Torricelli's Law, water drains through the hole at the rate of $dV/dt = -m\sqrt{D}$, where m is a positive constant. Find the rate at which the water level is decreasing if the bowl is generated by rotating around the y-axis the graph of **a.** $y = cx^2$ and **b.** $y = cx^4$, $c > 0$. Which of the two rates do you think can be used as a "water clock," and why?

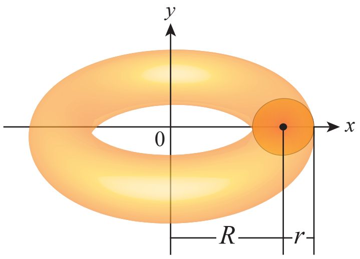
83. Although we typically work with circular cones, in a more general sense, cones don't have to be circular. In fact, if C is a simple closed curve in a plane S, and M is a point not in the same plane, then the solid generated by a line passing through M and moving along C, is said to be a cone (the region bounded by C is the base, M is the vertex of the cone, the distance between M and S is its height). Use the results and methods of this section to prove that the volume of a cone of base area B and height h is $V = \frac{1}{3}Bh$. (**Hint:** Mimic Example 2, using the fact that by similarity, in the general case, we have $\frac{A(y)}{B} = \frac{y^2}{h^2}$, that is, $A(y) = \frac{B}{h^2} \cdot y^2$.)



84.* Consider the circle in the xy-plane defined by the following equation.

$$(x-R)^2 + y^2 = r^2 \quad (R > r)$$

By rotating the region bounded by this circle around the y-axis, prove that the volume of the generated solid, a torus, is $V = 2\pi^2 Rr^2$.



85.* Rotate about the x-axis the region between the graphs of $y = \sqrt{4-x}$ and $y = c$ $(0 \le c \le 2)$ over the interval $[0,4]$. Find the value of c that minimizes the volume of the resulting solid. For what c-value is the volume maximal?

6.1 Technology Exercises

86. Use a graphing utility to repeat Exercise 85 for the region bounded by $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$ rotating about the line $y = c$ $(0 \le c \le 1)$.

87.* With the aid of a graphing utility, revisit Exercise 81, assuming this time that the angle between the can's axis of symmetry and the horizontal direction is 10°.

88–92 Use a graphing utility to find (or approximate) the volume of the solid generated by rotating the region bounded by the graphs of the given equations about the indicated axis.

88. $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = -1$, $x = 1$; about the x-axis

89. $y = \arcsin x$, $y = 0$, $x = 0$, $x = 1$; about the x-axis

90. $y = e^x$, $x = 0$, $y = 1$, $y = e^2$; about the y-axis

91. $y = \dfrac{1}{\log x}$, $y = 0$, $x = 10$, $x = 100$; about the x-axis

92. $x = e^{1-y^2}$, $y = 0$, $y = 1$, $x = 0$; about the y-axis

93–94 Use a graphing utility to sketch the region bounded by the graphs of the given equations. Approximate the intersection points, then find an approximation for the volume of the solid generated by rotating the region about the x-axis.

93. $y = x\cos^2 x$, $y = 2x(2-x)$

94. $y = x + 3$, $y = e^x + e^{-x} - 2$