Example 7 **Using the Fundamental Theorem of Calculus, Part II, to Evaluate a Definite Integral**

Recall that the difference between the functions F and G in Example 4 was determined to be $\int_5^{10} \frac{x-2}{x+1}\,dx$. Evaluate this integral.

Solution

As always, an antiderivative of the integrand will make the evaluation of the integral an easy task. In this case, though, it will take a bit more thought to arrive at an antiderivative—it's not immediately clear what sort of function has a derivative of $(x-2)/(x+1)$.

We will learn many techniques for systematically developing antiderivatives in coming sections, but in this case rewriting the integrand as follows will suffice.

$$\frac{x-2}{x+1} = \frac{(x+1)-3}{x+1} = 1 - \frac{3}{x+1}$$

The same result can be obtained by dividing $x-2$ by $x+1$.

$$\begin{array}{r|l} & 1 \\ x+1) & x-2 \\ & -(x+1) \\ \hline & -3 \end{array}$$

$$\frac{x-2}{x+1} = 1 - \frac{3}{x+1}$$

Also, since $\frac{d}{dx}\ln(x+1) = \frac{1}{x+1}$ (you should verify this),

$$\begin{aligned} \int_5^{10} \frac{x-2}{x+1}\,dx &= \int_5^{10}\left(1-\frac{3}{x+1}\right)dx \\ &= \left[x - 3\ln(x+1)\right]_5^{10} \\ &= (10-3\ln 11)-(5-3\ln 6) \\ &= 5-3\ln 11+3\ln 6 \\ &= 5+3\ln\frac{6}{11}. \end{aligned}$$

5.3 Exercises

1–8 Find every point c in the given interval at which $f(x)$ takes on its average value.

1. $f(x) = x^3$; $[0,2]$

2. $f(x) = \frac{x(6-x)}{2}$; $[0,6]$

3. $f(x) = \frac{-x^4}{4} + 4$; $[-2,2]$

4. $f(x) = e^x$; $[0,1]$

5. $f(x) = \sin x$; $[0,\pi]$

6. $f(x) = x - \sqrt{x+1}$; $[0,8]$

7. $f(x) = \csc^2 x$; $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

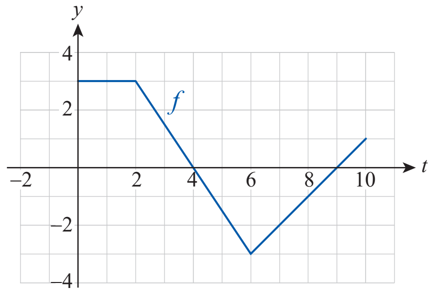
8. $f(x) = \frac{x^2+2}{x^2}$; $[1,3]$

9–10 Let $F(x)=\int_0^x f(t)\,dt$. Use the graph of f to answer the questions. (Note that the graph in Exercise 10 consists of linear and parabolic pieces.)

9. **a.** Evaluate $F(2)$, $F(4)$, $F(6)$, $F(8)$, and $F(10)$.

b. Give a formula for $F(x)$. (**Hint:** It will be a piecewise-defined function.)

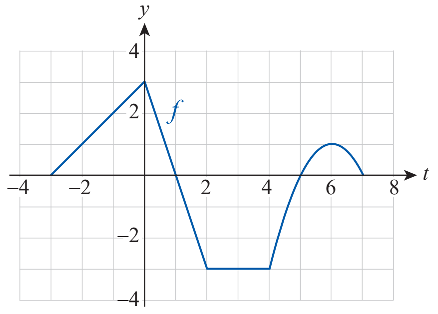
c. Sketch the graph of $F(x)$.



10. **a.** Evaluate $F(0)$, $F(2)$, $F(4)$, and $F(7)$.

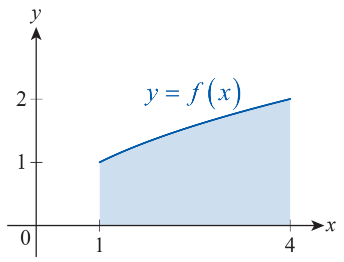
b. Give a formula for $F(x)$. (**Hint:** It will be a piecewise-defined function.)

c. Sketch the graph of $F(x)$.

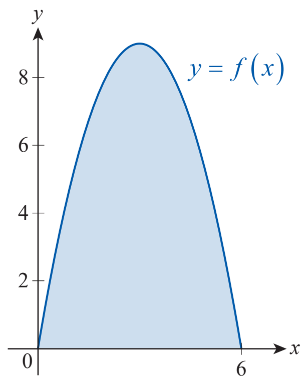


11–16 Find the area between the graph of $f(x)$ and the x-axis on the indicated interval.

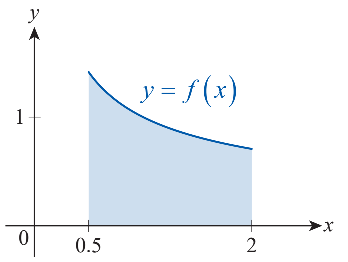
11. $f(x)=\sqrt{x}$ on $[1,4]$



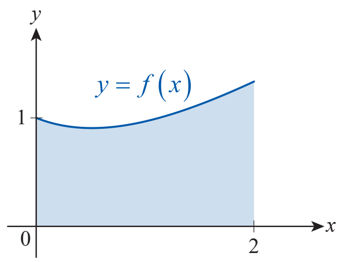
12. $f(x)=6x-x^2$ on $[0,6]$



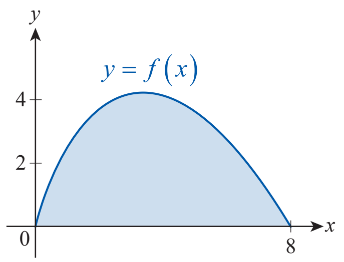
13. $f(x)=\dfrac{1}{\sqrt{x}}$ on $[0.5,2]$



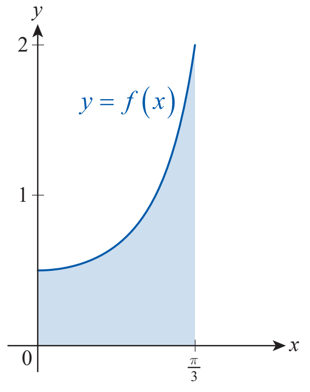
14. $f(x)=e^{-x}+0.6x$ on $[0,2]$



15. $f(x)=-2.5x^{4/3}+5x$ on $[0,8]$



16. $f(x)=0.5\sec^2 x$ on $\left[0,\dfrac{\pi}{3}\right]$



17–32 Use Part I of the Fundamental Theorem of Calculus to find the derivative of the given function.

17. $F(x)=\int_0^x \frac{1}{3}\left(t^2+\sqrt{t}\right)dt$

18. $F(x)=\int_{1/2}^x \ln s\, ds$

19. $G(x)=\int_{-4}^x \frac{t^4}{t^4+4}\,dt$

20. $G(x)=\int_2^x \sqrt[3]{u^2-u}\,du$

21. $y=\int_x^1 \sin\sqrt{t+1}\,dt$

22. $y=\int_x^0 t\arccos t\,dt$

23. $y=\int_{-5}^{3x}\left(t^2+3\right)e^{t-2}\,dt$

24. $y=\int_0^{x^2}\sec^{2/3}\sqrt{t}\,dt$

25. $y=\int_0^{\sin x}\left(t^2+e^t\right)dt$

26. $y=\int_{\sqrt{x}}^1 \log t\,dt$

27. $y=\int_x^{\pi x}\sin t\,dt$

28. $y=\int_{\sqrt{x}}^{x^2}\cos\left(z^2\right)dz$

29. $F(x)=\int_0^{\cos^{-1}x}\sqrt{1+\sqrt{1+\sec^2 t}}\,dt$

30. $G(x)=\int_{x-c}^{x+c}\sin t\,dt$

31. $H(x)=\int_{\ln x}^x \ln t\,dt$

32. $K(x)=\int_x^{x^2}\sqrt{1+t^4}\,dt$

33–38 Find a formula for $F(x)$ that is free of the integral symbol. Then differentiate it to verify Part I of the Fundamental Theorem of Calculus.

33. $F(x)=\int_1^x 2\,dt$

34. $F(x)=\int_{-3}^x (5-t)\,dt$

35. $F(x)=\int_x^1\left(t^2+t\right)dt$

36. $F(x)=\int_x^8 \frac{w+2}{\sqrt[3]{w}}\,dw$

37. $F(x)=\int_1^{\sqrt{x}}\frac{1}{s^2}\,ds$

38. $F(x)=\int_0^{\tan x}\left(1+u^2\right)du$

39–65 Use Part II of the Fundamental Theorem of Calculus to evaluate the definite integral.

39. $\int_{-2}^4 (-5)\,dx$

40. $\int_0^{1/\pi} 3\pi^2\,dx$

41. $\int_2^9 (4x+3)\,dx$

42. $\int_{-2.5}^6 (1-5u)\,du$

43. $\int_{-2}^4\left(1.5x^2-x+3\right)dx$

44. $\int_0^3 (5s-1)(2+s)\,ds$

45. $\int_1^7\left(2.4x^3-4x^2+1\right)dx$

46. $\int_{-1}^1\left(2x^2+1\right)^2 dx$

47. $\int_1^2\left(1-\frac{2}{x}\right)dx$

48. $\int_1^3\left(\frac{1}{x^2}+\frac{2}{x}+3\right)dx$

49. $\int_{-2}^{-1}\frac{2x^5-4x^2}{x^3}\,dx$

50. $\int_0^3 \frac{2x^2-\sqrt{x}}{4}\,dx$

51. $\int_1^2\left(x\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx$

52. $\int_2^4 \frac{5x^2-3x+2}{\sqrt{x}}\,dx$

53. $\int_{1/8}^1\left(2\sqrt[3]{t}-\sqrt[3]{\frac{2}{t}}\right)dt$

54. $\int_0^1 \frac{x+3\sqrt{x}}{\sqrt[5]{x}}\,dx$

55. $\int_0^{\pi/2}\left(\frac{\sin x}{2}-\sqrt{x}\right)dx$

56. $\int_0^{\pi/3}\frac{2}{\cos^2\theta}\,d\theta$

57. $\int_{\sqrt{2}/2}^1 \frac{3}{\sqrt{1-t^2}}\,dt$

58. $\int_{\sqrt{3}/3}^1 \frac{-5}{1+x^2}\,dx$

59. $\int_0^{\pi/3}\left(e^x+\sec x\tan x\right)dx$

60. $\int_{-3}^3 2^x\,dx$

61. $\int_{-2}^4 \left|x(x-2)\right|dx$

62. $\int_1^3 f(x)\,dx$, where $f(x)=\begin{cases}\sin\frac{\pi x}{4} & \text{if } 0<x\le 2\\ (x-2)^2+1 & \text{if } 2<x\le 3\end{cases}$

63. $\int_{\pi/4}^{3\pi/4}(1-\csc\theta\cot\theta)\,d\theta$

64. $\int_{\pi/4}^{\pi/2}\frac{2}{1-\cos^2 x}\,dx$

65. $\int_{-1}^1 g(x)\,dx$, where $g(x)=\begin{cases}\sqrt{x+1} & \text{if } -1<x\le 0\\ e^x & \text{if } 0<x\le 1\end{cases}$

66–69 Recognize the given limit as a Riemann sum of a function over an interval and then use the Fundamental Theorem of Calculus to evaluate it.

66. $\lim_{n\to\infty}\sum_{i=1}^n \frac{\sqrt{i}}{n^{3/2}}$

67. $\lim_{n\to\infty}\sum_{i=1}^n \frac{2}{n}\left(\frac{2i}{n}\right)^4$

68. $\lim_{n\to\infty}\sum_{i=1}^n \frac{\pi}{2n}\cos\frac{\pi i}{2n}$

69. $\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{e-1}{n+i(e-1)}$

70–78 Find the area of the region between the graph of the given function and the *x*-axis on the indicated interval.

70. $y = \cos x$ on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

71. $y = -x^3$ on $[-2, 2]$

72. $y = -x^2 + 1$ on $[-1, 2]$

73. $y = \frac{1}{x}$ on $\left[\frac{1}{e}, e^2\right]$

74. $y = -2|x-3| + 6$ on $[0, 10]$

75. $y = -x^3 + 7x^2 - 10x$ on $[0, 6]$

76. $y = 2\sqrt{x} - x$ on $[0, 9]$

77. $y = \frac{1-2x}{2x+1}$ on $[0, 1]$

78. $y = x^4 - x^2$ on $[-1, 1]$

79–87 Use the method of Example 7 to evaluate the definite integral.

79. $\int_3^4 \frac{x}{x-2}\,dx$

80. $\int_5^7 \frac{x+5}{x-4}\,dx$

81. $\int_0^{e-1} \frac{2x-5}{x+1}\,dx$

82. $\int_{-3/2}^0 \frac{3x-1}{2x+4}\,dx$

83. $\int_0^1 \frac{3x^2+4}{x^2+1}\,dx$

84. $\int_0^2 \frac{5x^2-1}{2x^2+4}\,dx$

85. $\int_4^6 \frac{3x^2+2x-9}{x^2-3}\,dx$

86. $\int_0^2 \frac{2x^2+4x+11}{x^2+x+5}\,dx$

87. $\int_{-1}^1 \frac{x^3+5x^2+4x+1}{x^3+2x^2+1}\,dx$

88–90 The function $v(t)$ gives the velocity, in units per second, of a particle moving along the *x*-axis, having started from the origin. Find **a.** the position of the particle at $t = t_0$ and **b.** the total distance traveled by the particle in the time interval $[0, t_0]$.

88. $v(t) = 1 - (t-1)^2; \quad t_0 = 4$

89. $v(t) = \frac{t-1}{2(t+1)}; \quad t_0 = 3$

90. $v(t) = t(t-3)(t-5); \quad t_0 = 6$

91. Find a formula for $f(x)$ if $\int_0^x f(t)\,dt = \sin 2x + x$.

92.* Repeat Exercise 91 if $\int_0^{x^2} f(t)\,dt = x^3$.

93.* Let $f(x) = x - [\![x]\!]$ and $F(x) = \int_0^x f(t)\,dt$. Prove that F is continuous, briefly discuss its graph, and sketch it on paper.

94. Show that the piecewise-defined function $F(x) = \begin{cases} -\frac{1}{2}x^2 & \text{if } x \le 0 \\ \frac{1}{2}x^2 & \text{if } x > 0 \end{cases}$ is an antiderivative of $f(x) = |x|$. Then find an easier formula for $F(x)$ and use the Fundamental Theorem of Calculus to evaluate $\int_a^b |x|\,dx$.

95. Write a paragraph entitled "Differentiation and Integration as Inverse Operations." Quote the Fundamental Theorem of Calculus and include concrete examples.

96.* Use the properties of the definite integral to show directly that if $f(x)$ is integrable on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on the same interval. (**Hint:** Argue that there is an M so that $|f(x)| \le M$ on $[a, b]$ and use the result of Exercise 103 of Section 5.2.)

97. Let $l(x)$ be defined as the integral function of $1/x$, that is, $l(x) = \int_1^x (1/t)\,dt$. Show that $l(x) = \ln x$. (**Hint:** See the discussion in Example 4.)

98.* Use the definition from Exercise 97 to show the following well-known property of logarithms: For positive $a, b \in \mathbb{R}$, $l(a \cdot b) = l(a) + l(b)$. (**Hint:** Use the definition to show that $l'(ax) = l'(x)$, which implies $l(ax) = l(x) + C$ for some constant C. Argue that $l(a) = C$. Finally, let $x = b$.)

99.* Use the definition from Exercise 97 to show that $l(1/x) = -l(x)$.

100. Taking a cue from Exercises 97–99, let

$$\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0$$

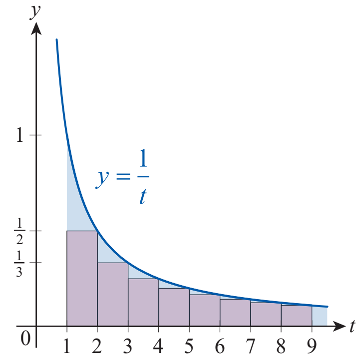
be the definition of the natural logarithm function—that is, let all our knowledge of the natural logarithm function be determined by this particular definite integral. Note that, by the Fundamental Theorem of Calculus (which applies, since $1/t$ is continuous on the interval $0 < t < \infty$), the natural logarithm is a differentiable function and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

a. Prove that $\ln 1 = 0$ and that $\lim_{x\to\infty} \ln x = \infty$. (**Hint:** Construct a Riemann sum based on the given figure to show that

$$\int_1^x \frac{1}{t}\,dt > \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \cdots$$
$$> \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

for sufficiently larger x.)



Note that these two facts, along with the fact that $\ln x$ is a continuous function, implies that $\ln x$ takes on every positive real value over the interval $1 < x < \infty$ (and also, given the result of Exercise 99, every negative real value over the interval $0 < x < 1$).

b. Prove that $\ln x$ is one-to-one and hence has an inverse function. (**Hint:** Prove that $\ln x$ is strictly increasing.) Given this fact, define e^x to be the inverse of $\ln x$; that is, define e^x by $e^x = \ln^{-1} x$. In particular, define $e = \ln^{-1}(1)$.

c. Use L'Hôpital's Rule to prove $\lim_{u\to 0} \frac{u}{\ln(1+u)} = 1$.

d. Use the result from part c. to prove $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$. (**Hint:** Let $u = e^h - 1$ and note that $u \to 0$ as $h \to 0$.)

101. Archimedes (287–212 BC) discovered that the area under a parabolic arch is two-thirds the length of the base times its height. Sketch the graph of $y = h - ax^2$, the general parabolic arch with vertex at $(0, h)$ and use the FTC to verify Archimedes' formula. (Note the interesting parallel between Archimedes' formula and that of the area of an isosceles triangle of the same base and height.)

102. The marginal cost of production of baby toys at a small company has been determined to be $C'(x) = 200/\left(3\sqrt[3]{x}\right)$ dollars. How much will it cost to increase production from 400 to 500 toys?

Concept Check

103–108 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

103. If $f(x)$ is continuous on $[a,b]$ and $c \in [a,b]$ is the point guaranteed by the Mean Value Theorem for Definite Integrals, then $y = f(x)$ and the constant function $y = f(c)$ both have the same definite integral on $[a,b]$.

104. When evaluating a definite integral using the Fundamental Theorem of Calculus, we can use *any* of the antiderivatives of the integrand.

105. If $f(x)$ is a continuous, odd function on $\mathbb{R}$ and $F(x) = \int_{-a}^{x} f(t)\,dt$ for some $a > 0$, then $F(x)$ has a zero at $x = a$.

106. If $f(x)$ is integrable on $\mathbb{R}$, then $\int_a^x f(t)\,dt$ and $\int_b^x f(t)\,dt$ have the same derivative for all $a, b \in \mathbb{R}$.

107. $\frac{d}{dx}\int_a^{x^3} (t+1)^3\,dt = \left(x^3+1\right)^3$

108. If f is continuous on $[a,b]$ and F is any antiderivative of f, then the area of the region bounded by the graph of f and the x-axis is $F(b) - F(a)$.