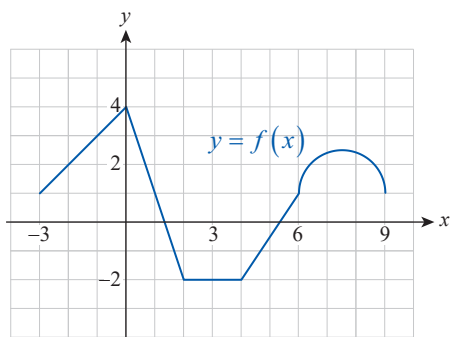


5.2 Exercises

1. Use the given graph along with appropriate formulas from geometry to evaluate each of the indicated definite integrals. (Note that the graph of f consists of linear pieces and a semicircle.)

a. $\int_{-3}^1 f(x) dx$ b. $\int_{-3}^9 f(x) dx$
 c. $\int_0^6 |f(x)| dx$ d. $\int_0^9 [f(x) - 2] dx$



2. Calculate the total distance traveled by the stone of Example 1 by evaluating $\int_0^4 |v(t)| dt$.
3. Suppose that in Example 1, George shoots the stone upward while standing near the edge of a deep canyon, and this time pulls the slingshot a bit harder, achieving a velocity function of $v(t) = 80 - 32t$ ft/s. What is the height of the stone, relative to its initial height, at $t = 4$ seconds? How about at $t = 6$ seconds? 10 seconds?

4–13 Use the given partition and sample points to approximate the definite integral of $f(x)$ on the indicated interval. (Note that the subintervals do not always have to be of equal width, and the sample points may be unevenly spaced.)

4. $f(x) = \frac{1}{3}x + 1$, $x_0 = 0 < 1 < 2 < 3 < 4 < 5 < 6 = x_6$, $x_i^* = x_i$
5. $f(x) = x^2 + x + 2$, $x_0 = -1 < 0 < 1 < 2 < 3 = x_4$, $x_i^* = x_{i-1}$
6. $f(x) = -x - \frac{3}{2}$, $x_0 = -2 < -1.5 < -0.9 < 0 < 1 = x_4$, $x_1^* = -1.8$, $x_2^* = -1$, $x_3^* = -0.4$, $x_4^* = 0.5$
7. $f(x) = \frac{1}{x^2}$, $x_0 = 1 < 2 < 3 < 4 = x_3$, $x_i^* = \frac{x_{i-1} + x_i}{2}$
8. $f(x) = \frac{1}{1+x^2}$, $x_0 = -3 < -2 < -1 < 0 < 1 < 2 < 3 = x_6$, $x_i^* = \frac{x_{i-1} + x_i}{2}$
9. $f(x) = x^3 - x$, $x_0 = 0 < 0.3 < 0.5 < 1 < 1.5 = x_4$, $x_1^* = 0.25$, $x_2^* = 0.5$, $x_3^* = 1$, $x_4^* = 1.2$
10. $f(x) = \sin x$, $x_0 = 0 < \frac{\pi}{6} < \frac{\pi}{4} < \frac{\pi}{3} < \frac{\pi}{2} < \frac{2\pi}{3} < \frac{3\pi}{4} < \frac{5\pi}{6} < \pi = x_8$, $x_i^* = x_{i-1}$
11. $f(x) = \ln(x+1)$, $x_0 = -0.5 < 1 < 2 < 2.5 = x_3$, $x_1^* = 0$, $x_2^* = e - 1$, $x_3^* = 2$
12. $f(x) = 10^{-x}$, $x_0 = 0 < 0.05 < 0.15 < 1 = x_3$, $x_1^* = 0.01$, $x_2^* = 0.1$, $x_3^* = 1$
13. $f(x) = \sqrt{x}$, $x_0 = 0 < \frac{1}{25} < \frac{4}{25} < \frac{9}{25} < \frac{16}{25} < 1 = x_5$, $x_i^* = x_i$

14–27 Use the concept of the definite integral to find the total area between the graph of $f(x)$ and the x -axis, by taking limits of the associated Riemann sums. When setting up the Riemann sums, make your choice between the left-endpoint, right-endpoint, and midpoint strategies. (**Hint:** Extra care is needed on those intervals where $f(x) < 0$. Remember that the definite integral represents a signed area.)

14. $f(x) = 2x + 4$ on $[0, 2]$ 15. $f(x) = x - 1$ on $[0, 5]$
 16. $f(x) = \frac{3-x}{2}$ on $[0, 5]$ 17. $f(x) = x^2$ on $[1, 3]$

18. $f(x) = x^2 - 1$ on $[-1, 1]$

19. $f(x) = x^2 - 4x$ on $[0, 5]$

20. $f(x) = \frac{x^2}{2} + 2$ on $[-2, 2]$

21. $f(x) = 3x^2 - 3$ on $[-1, 1]$

22. $f(x) = x^2 - 2x - 3$ on $[-1, 4]$

23. $f(x) = x^3$ on $[0, 1]$

24. $f(x) = 4x^3 - 32$ on $[0, 2]$

25. $f(x) = x^3 + 3x^2 + 1$ on $[0, 3]$

26. $f(x) = \begin{cases} 1 - (x-1)^2 & \text{if } 0 \leq x \leq 3 \\ x-6 & \text{if } 3 \leq x \leq 4 \end{cases}$

27. $f(x) = \begin{cases} x^3 & \text{if } 0 \leq x \leq 2 \\ 8x - 2x^2 & \text{if } 2 \leq x \leq 4 \end{cases}$

28. Generalize Exercise 13 to n subintervals and find the definite integral $\int_0^1 \sqrt{x} dx$ by letting $n \rightarrow \infty$. (**Hint:** Let $x_i^* = i^2/n^2$.)

29. Use the same approach as in Exercise 28 to find $\int_0^2 \sqrt[3]{x} dx$. (**Hint:** Let $x_i^* = 2i^3/n^3$.)

30–33 Express the integral as a limit of Riemann sums. (Do not attempt to evaluate the limit.)

30. $\int_1^3 \frac{1}{x} dx$

31. $\int_0^4 (x^2 - \log_2 x) dx$

32. $\int_{-a}^a \frac{1}{x^2 + 1} dx$

33. $\int_2^b \sqrt[4]{x} dx$

34–45 Sketch the region whose (signed) area is represented by the definite integral, and then use appropriate formulas from geometry to evaluate the integral.

34. $\int_{-1}^5 3 dx$

35. $\int_{2.5}^{12} (-2) dx$

36. $\int_4^2 (1-x) dx$

37. $\int_8^3 \left(4 - \frac{1}{2}x\right) dx$

38. $\int_0^4 |2x-3| dx$

39. $\int_{-1}^5 (5 - |2x|) dx$

40. $\int_0^{10} (|x-2| - |7-x|) dx$

41. $\int_{-5}^0 \sqrt{25-x^2} dx$

42. $\int_{-a/2}^a \sqrt{a^2 - x^2} dx, a > 0$

43. $\int_{-2}^5 (2 - \lfloor x \rfloor) dx$

44. $\int_{-3}^8 \lfloor 3x-1 \rfloor dx$

45. $\int_{-4}^6 (x - \lfloor x \rfloor) dx$

46. Use Riemann sums resulting from midpoint estimates to prove $\int_a^b x dx = (b^2 - a^2)/2$. (**Hint:** Notice that after using $(b+a)(b-a) = b^2 - a^2$, each Riemann sum becomes a collapsing sum.)

47. Provide an alternate proof for Exercise 46 by making a sketch and using areas of triangles.

48.* Mimic the argument used in Exercise 46, but using

$$x_i^* = \sqrt{(x_{i-1}^2 + x_{i-1}x_i + x_i^2)}/3, \text{ to prove the formula } \int_a^b x^2 dx = (b^3 - a^3)/3.$$

49. The Dirichlet function is defined as follows.

$$\xi(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that $\xi(x)$ is not integrable. (**Hint:** For a given n , form a Riemann sum by choosing each sample point x_i^* to be rational, then see what happens if each x_i^* is irrational. Use your observation to argue that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \text{ does not exist.}$$

50. Prove that the function $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not integrable on $[0, 1]$.

(**Hint:** By examining the first term of each R_n , show that $\lim_{n \rightarrow \infty} R_n$ does not exist.)

51. Repeat Exercise 50 for $g(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ on $[0, 1]$.

(**Hint:** Show that arbitrarily large Riemann sums can be constructed by choosing appropriate x_i^* 's.)

52–59 Decide whether the function is integrable on the indicated interval. If not, say why. (Do not evaluate the integral.)

52. $f(x) = \frac{1}{\sqrt{x+2}}$ on $[-1, 1]$

53. $g(x) = \frac{2}{x}$ on $[-2, 2]$

54. $h(x) = \frac{-3}{x-1}$ on $[0, 5]$

55. $F(x) = \frac{x}{|x|}$ on $[-3, 4]$

56. $G(x) = x \cdot \lfloor x \rfloor$ on $[-2, 2]$

57. $H(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ on $[-1, 1]$

$$58. u(x) = \begin{cases} \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \text{ on } [-1, 1]$$

$$59. v(x) = \begin{cases} 3.14 & \text{if } x \text{ is rational} \\ \pi & \text{if } x \text{ is irrational} \end{cases} \text{ on } [0, 2]$$

60–65 Match the given property of the definite integral to the relevant illustration (labeled A–F).

$$60. \int_a^b k \, dx = k(b-a) \quad (\text{Property 3})$$

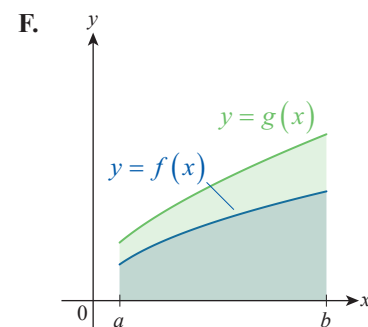
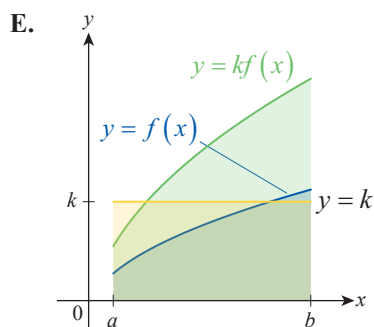
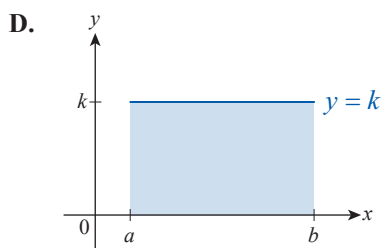
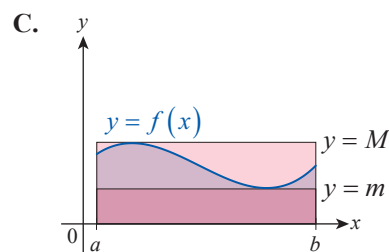
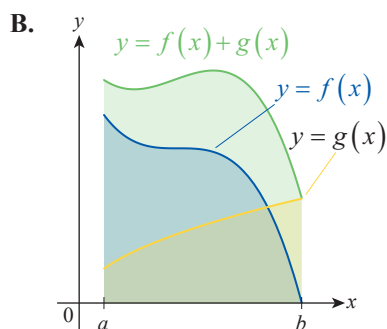
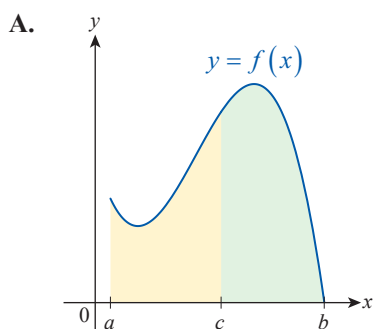
$$61. \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \quad (\text{Property 4})$$

$$62. \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \quad (\text{Property 5})$$

$$63. \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx \quad (\text{Property 6})$$

$$64. \text{ If } f(x) \leq g(x) \text{ on } [a, b], \text{ then } \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx. \quad (\text{Property 7})$$

$$65. \text{ If } m = \min_{a \leq x \leq b} f(x) \text{ and } M = \max_{a \leq x \leq b} f(x), \text{ then } m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a). \quad (\text{Property 8})$$



66–75 Use the properties of the definite integral to find the given integral, if possible, given that $\int_a^b f(x) \, dx = 3$, $\int_c^b f(x) \, dx = -1$, and $\int_a^b g(x) \, dx = -5$.

$$66. \int_a^b [f(x) - g(x)] \, dx$$

$$67. \int_a^c [2f(x) + 1] \, dx$$

$$68. \int_c^a 10f(x) \, dx$$

$$69. \int_a^a f(x)g(x) \, dx$$

$$70. \int_a^b \left[4f(x) + \frac{g(x)}{10} \right] \, dx$$

$$71. \int_b^a \frac{\sqrt{2}}{2} g(x) \, dx$$

$$72. \int_a^b [f(x) + 2g(x) - 2] \, dx$$

$$73. \int_a^b [f(x)]^2 \, dx$$

$$74. \int_a^b \frac{5}{g(x)} \, dx$$

$$75. \int_a^b \left[\frac{f(x)}{3} - \pi g(x) \right] \, dx$$

76–83 Use the results from Exercises 46 and 48, along with the properties of the definite integral and formulas from geometry, to evaluate the given integral.

$$76. \int_0^2 (3x-1) dx \qquad 77. \int_{\sqrt{2}}^{-1} \left(1 - \frac{\sqrt{2}}{2}x\right) dx$$

$$78. \int_{-1}^4 (x^2 + 5) dx \qquad 79. \int_1^4 (2x^2 - x) dx$$

$$80. \int_0^3 \left(t^2 + \frac{t}{4} + 4\right) dt \qquad 81. \int_0^1 (2\sqrt{x} + x) dx$$

$$82. \int_2^0 \left(\frac{\sqrt[3]{x}}{4} - x^2\right) dx \qquad 83. \int_{-2}^2 (u - 3\sqrt[3]{u}) du$$

84. Suppose that f is an even function, g is odd, and both are integrable on $[-a, a]$. Use the properties of the definite integral to prove the following statements.

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \text{and} \quad \int_{-a}^a g(x) dx = 0$$

85–90 Suppose that f is an even function, g is odd, both are integrable on $[-2, 2]$, and we know that $\int_0^2 f(x) dx = 1$, while $\int_0^2 g(x) dx = 2.5$. If possible, find the integral.

$$85. \int_{-2}^2 [f(x) + g(x)] dx$$

$$86. \int_{-2}^2 [2f(x) - 3g(x)] dx$$

$$87. \int_{-2}^2 g(x) dx \qquad 88. \int_{-2}^2 f(x)g(x) dx$$

$$89. \int_{-1}^1 [f(x)]^2 dx \qquad 90. \int_0^2 |g(x)| dx$$

91. Use Property 8 of the definite integral to prove the validity of the following upper and lower estimates:
 $12 \leq \int_0^4 \sqrt{x^2 + 9} dx \leq 20$.

92–96 Use an argument similar to the one you gave in Exercise 91 to give upper and lower estimates for the given definite integral.

$$92. \int_{-1}^4 \sqrt{5+x} dx \qquad 93. \int_2^3 \sqrt{3-x} dx$$

$$94. \int_4^5 \frac{1}{x-2} dx \qquad 95. \int_0^6 \left(\frac{x^2}{32} - \frac{x}{4} + \frac{3}{2}\right) dx$$

$$96. \int_1^{\sqrt{3}} \arctan x dx$$

97. Use Property 7 of the definite integral to prove the following inequalities.

$$\text{a. } \int_0^1 \sqrt{1-x} dx \leq \int_0^1 \sqrt{1-x^2} dx$$

$$\text{b. } \int_0^{\pi/2} \cos x dx \leq \int_0^{\pi/2} \frac{\sin x}{x} dx$$

98. Prove Property 4 of the definite integral. (**Hint:** Write a typical Riemann sum for f on $[a, b]$; use the Constant Multiple Rule for Finite Sums, followed by properties of limits.)

99. Prove Property 6 of the definite integral

in general; that is, prove that the property $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$ holds

irrespective of the order of the points a , b , and c .

(**Hint:** The standard case of $a < c < b$ is discussed in the text. To start you off with the remaining cases, assume, for example, that $a < b < c$. By an argument analogous to the one given in the text, we see that

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx.$$

Observe by Property 2 that $\int_b^c f(x) dx = -\int_c^b f(x) dx$, and rearrange the terms. Handle the remaining cases in a similar fashion.)

100. Prove Property 7 of the definite integral. (**Hint:** For a particular partition of $[a, b]$ and choice of sample points, argue that $\sum_{i=1}^n f(x_i^*) \Delta x \leq \sum_{i=1}^n g(x_i^*) \Delta x$, and take the limits as $n \rightarrow \infty$.)

101. Use Property 7 to prove that the definite integral of a nonnegative function is nonnegative: If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) dx \geq 0$. Then state and prove the analogous statement for nonpositive functions.

102. Prove Property 8 of the definite integral. (**Hint:** Use Property 7 with the constant function $g(x) = M$. The other inequality can be handled in a similar manner.)

103. Use Properties 4 and 7 to prove the following statement: If f is integrable on $[a, b]$, then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx. \quad (\text{Hint: Let } k = 1 \text{ or } k = -1 \text{ so that } \left| \int_a^b f(x) dx \right| = k \cdot \int_a^b f(x) dx. \text{ Use the fact that } k \cdot f(x) \leq |f(x)|, \text{ along with Properties 4 and 7.)}$$

104–115 Find the average value of the function over the given interval. (**Hint:** Instead of using Riemann sums, try using the results from Exercises 46 and 48 along with formulas from geometry and the properties of the definite integral.)

$$104. f(x) = 3x - 1 \text{ on } [0, 4]$$

$$105. g(x) = -1 - \frac{1}{2}x \text{ on } [-2, 2]$$

$$106. h(x) = x^2 - 2 \text{ on } [-1, 5]$$

$$107. F(x) = -3x^2 + 7x + 12 \text{ on } [-2, 3]$$

$$108. G(x) = 9x - x^3 \text{ on } [-4, 4]$$

109. $H(x) = x^3 - 2x^2 - 1$ on $[0, 2]$

110. $k(x) = |x - 4| - 2$ on $[0, 7]$

111. $m(x) = |x| + |x + 1|$ on $[-3, 2]$

112. $u(x) = \sqrt{1 - (x - 1)^2}$ on $[0, 2]$

113. $v(x) = \llbracket x \rrbracket$ on $\left[\frac{1}{2}, 3\right]$

114. $t(x) = \sqrt{x} - 1$ on $[0, 5]$

115. $w(x) = \sqrt[3]{x} - x$ on $[-1, 8]$

116–119 Recognize the given limit as a Riemann sum of a function over an interval and then use geometry to evaluate it.

116. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left(2 - \frac{i}{n}\right)$

117. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left(\frac{2i}{3n} + 4\right)$

118. $\lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} \left(\frac{2}{n} + \frac{4i}{n^2}\right)$

119. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \sqrt{4 - \left(\frac{2i}{n}\right)^2}$

120. Prove that if $f(x)$ is an increasing nonnegative function on $[a, b]$, then for every n ,

$L_n \leq \int_a^b f(x) dx \leq R_n$. Then state and prove the analogous statement for a decreasing function $g(x)$ on the same interval.

121. Prove that L_n corresponding to $f(x)$ of Exercise 120 is increasing as $n \rightarrow \infty$, while R_n is decreasing. Then state and prove the analogous statement for $g(x)$.

122. Use geometry and a fundamental trigonometric identity to find $\int_0^\pi \sin^2 x dx$. (**Hint:** Start out by comparing the given integral with $\int_0^\pi \cos^2 x dx$.)

123. Use the result of Exercise 122 to evaluate

$$\int_0^\pi (2 \sin^2 x + x^2 - 3x) dx.$$

124.* Suppose that the nonnegative function $R(x)$ has the property that $R(x) = 0$ whenever x is rational. If R is integrable on the interval $[a, b]$, prove that $\int_a^b R(x) dx = 0$.

Concept Check

125–130 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

125. If f and g are both integrable on $[a, b]$, then $\int_a^b f(x) \cdot g(x) dx = \int_a^b f(x) dx \cdot \int_a^b g(x) dx$.

126. The integral $\int_a^b f(x) dx$ is numerically equal to the area between the graph of $f(x)$ and the x -axis.

127. A Riemann sum for $f(x)$ on $[a, b]$ can be based upon a division of $[a, b]$ into subintervals of unequal width.

128. If $|f(x)|$ is integrable on $[a, b]$, then so is $f(x)$.

129. If $f(x)$ is positive and increasing on $[a, b]$, then $\int_a^b f(x) dx \geq f(a)(b - a)$.

130. If $\int_a^b f(x) dx < 0$, then $f(x) \leq 0$ on $[a, b]$.