Example 4 **Finding the Area Bounded by the *x*-Axis and the Graph of a Function**

Find the area under the graph of $f(x)=x^2$ and above the x-axis on the interval $[0,1]$.

Solution

For variety, we will construct an expression for U_n (the underestimate based on n subintervals) and evaluate $\lim_{n\to\infty} U_n$. In Exercise 77 you will show that the same answer is obtained by evaluating $\lim_{n\to\infty} O_n$.

If we divide $[0,1]$ into n subintervals of equal width, each one has width $\Delta x = 1/n$. The minimum value of f on $[x_{i-1}, x_i]$ occurs at $x_i^* = x_{i-1}$, that is, at the left endpoint of each subinterval. So $x_1^* = 0$, $x_2^* = 1/n$, $x_3^* = 2/n$, and in general $x_i^* = (i-1)/n$. So we have the following Riemann sum.

$$U_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2$$

We have already simplified the sum $\sum_{i=1}^{n}(i-1)^2$ in part c. of Example 3.

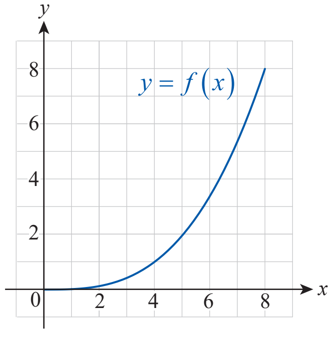
$$U_n = \frac{1}{n^3}\left(\frac{2n^3-3n^2+n}{6}\right) = \frac{2n^3-3n^2+n}{6n^3}$$

Hence, $\lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{2n^3-3n^2+n}{6n^3} = \frac{2}{6} = \frac{1}{3}$.

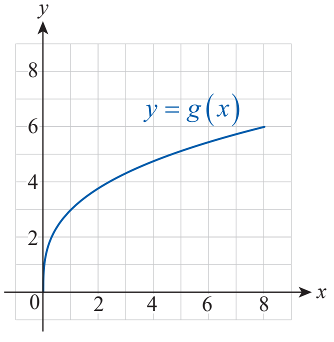
5.1 Exercises

1–2 Use $(O_4+U_4)/2$ to estimate the area under the graph of the function and above the x-axis on the interval $[0,8]$.

1. $f(x)=\left(\frac{x}{4}\right)^3$

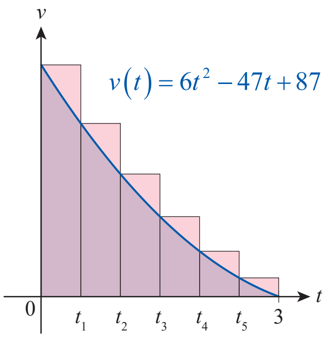


2. $g(x)=3\sqrt[3]{x}$



3–4. Repeat Exercises 1–2 using eight rectangles.

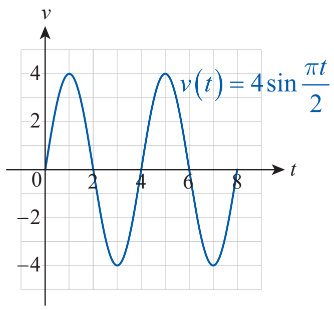
5. The figure below shows the upward velocity (in feet per second) of a model rocket during its rise. Use the method of Example 1 to estimate how high the rocket rose by calculating O_6. Is your estimate an over- or underestimate?



6. The velocity of an object undergoing simple harmonic motion is given by the graph below (time is measured in seconds, distance in feet). Using subintervals of width $\frac{1}{4}$,

a. give an overestimate for the total distance covered from $t = 1$ s to $t = 6$ s;

b. estimate the total displacement from $t = 1$ s to $t = 6$ s.



7. The given table contains the velocity data recorded by an automotive testing device during an acceleration test.

Time (s)	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
v (m/s)	3	6.6	9.8	13	16.1	19.1	21.6	23.8	25.8	27.6	28.5	29.1

a. Use 12 subintervals to give over- and underestimates of the distance covered by the car during the acceleration run (i.e., find O_{12} and U_{12}).

b. Approximate the above distance using 6 subintervals of equal width and choosing the midpoint of each as the sample point (we shall call the resulting quantity M_6).

c. Compare M_6 with $(O_{12}+U_{12})/2$. Which one is greater? Explain why this is the case.

8. In order to estimate the length of the runway, a passenger on an airplane jotted down some velocity data during takeoff from the on-board entertainment screen. From the resulting table given below, calculate $(O_8+U_8)/2$ to find his estimate.

Time (s)	6	12	18	24	30	36	42	48
v (mph)	30	79	115	150	180	204	223	230

9–14 Use four rectangles to estimate the area between the graph of the given function and the x-axis on the given interval. Construct three estimates for the function: the first using the left endpoints of the subintervals as the sample points, the second using the right endpoints of the subintervals, and the third using the midpoints of the subintervals. Can you tell which are guaranteed to be underestimates or overestimates? (**Hint:** Consider the increasing/decreasing and concavity features of the graph. It is helpful to make a sketch.)

9. $f(x)=\sqrt{x}$ on $[0,4]$

10. $f(x)=\dfrac{x^3}{16}$ on $[0,4]$

11. $f(x)=\dfrac{1}{x}$ on $[1,5]$

12. $f(x)=\sqrt{4-x^2}$ on $[-2,2]$

13. $f(x)=\cos\dfrac{x}{2}$ on $[0,\pi]$

14. $f(x)=e^{2-x}$ on $[0,2]$

15–24 Write the given sum using sigma notation.

15. $3+6+9+\cdots+99$

16. $1+2+9+28+\cdots+(25^3+1)$

17. $1+\dfrac{1}{4}+\dfrac{1}{9}+\dfrac{1}{16}+\cdots+\dfrac{1}{10,000}$

18. $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\cdots-\dfrac{1}{50}$

19. $a_{-3} + a_{-1} + a_1 + a_3 + a_5 + \cdots + a_{77}$

20. $b_0 + b_3 + b_6 + b_9 + b_{12} + \cdots + b_{297}$

21. $f\left(\frac{3}{n}\right) + f\left(\frac{6}{n}\right) + f\left(\frac{9}{n}\right) + \cdots + f\left(\frac{3(n-1)}{n}\right) + f(3)$

22. $g(c_0) + g(c_5) + g(c_{10}) + g(c_{15}) + \cdots + g(c_{650})$

23. $f(x_0^*)\Delta x + f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$

24. $s(t_1^*)\Delta t + s(t_2^*)\Delta t + s(t_3^*)\Delta t + \cdots + s(t_{n-1}^*)\Delta t$

25–30 Assuming that $\sum_{i=0}^{n} a_i = 36$ and $\sum_{i=0}^{n} b_i = 100$, find the given sum.

25. $\sum_{i=0}^{n} (b_i - a_i)$

26. $\sum_{i=0}^{n} (2a_i + 3b_i)$

27. $\sum_{i=0}^{n} (5b_i + 1)$

28. $\sum_{i=0}^{n} \left(\frac{a_i}{6} - \frac{b_i}{2}\right)$

29. $\sum_{j=0}^{n} \left(\frac{4a_j}{3} - \frac{b_j}{4} + 2\right)$

30. $\sum_{k=1}^{n} \left(\frac{4}{3} - 2a_k + \frac{b_k}{2}\right)$

31–42 Find the value of the sum. Use a summation formula when possible.

31. $\sum_{i=2}^{4} \frac{1}{i-1}$

32. $\sum_{j=-2}^{2} \sqrt{j+2}$

33. $\sum_{i=1}^{10} (5i - 2)$

34. $\sum_{i=1}^{n} (1 - 3i)$

35. $\sum_{j=1}^{n} \frac{4j + 5n}{2}$

36. $\sum_{k=1}^{n} \frac{6k^2 + 2k}{3}$

37. $\sum_{j=1}^{30} (2j^2 - 4j + 1)$

38. $\sum_{i=1}^{100} (2i-1)(3-i)$

39. $\sum_{j=1}^{n} (3j+1)^2$

40. $\sum_{i=1}^{n} \left(i^3 - 2i^2 + \frac{1}{n}\right)$

41. $\sum_{j=1}^{n} 2j^2 (j-2)$

42. $\sum_{k=0}^{n} k(k+1)(k+2)$

43–48 Write out the first few terms as well as the last few terms of the sum. Find a way to simplify and use your observation to evaluate the sum. (Sums of this type are called *collapsing sums*.)

43. $\sum_{i=1}^{10} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

44. $\sum_{k=3}^{n} \left[\frac{1}{k^3} - \frac{1}{(k+1)^3}\right]$

45. $\sum_{j=1}^{n} \left(\sqrt{j} - \sqrt{j+1}\right)$

46. $\sum_{k=1}^{n+1} \ln \frac{k}{k+1}$

47. $\sum_{j=2}^{n+3} \left(e^j - e^{j+1}\right)$

48. $\sum_{k=1}^{2n+1} \left[\sin k\pi - \sin((k+1)\pi)\right]$

49–52 A *geometric sum* (or *geometric progression*) is a sum of the following form.

$$a + ar + ar^2 + \cdots + ar^n = \sum_{i=0}^{n} ar^i, \quad r \neq 1$$

(Notice that each term is a constant multiple of the preceding term; this constant is called the *common ratio* and is denoted by r.)

Use the formula $\sum_{i=0}^{n} ar^i = a\frac{1-r^{n+1}}{1-r}$ to evaluate the sum.

49. $\sum_{i=0}^{10} 3^i$

50. $\sum_{j=0}^{8} 5\left(\frac{1}{2}\right)^j$

51. $\sum_{k=0}^{99} (-1)^k \left(\frac{2}{3}\right)^k$

52. $\sum_{n=0}^{1000} 4.9(-3.9)^n$

53. Prove the formula for the sum of the first $n + 1$ terms of a geometric progression given in the directions preceding Exercises 49–52. (**Hint:** Let S denote the sum, recognize $S - rS$ as a collapsing sum, evaluate, and solve for S.)

54–57 Sometimes, sums become easier to manage (the general term becomes simpler) after an appropriate shift in the index. For example, $\sum_{i=1}^{n+1}(i-1)^2$ can be rewritten as $\sum_{i=0}^{n} i^2$. Perform an appropriate shift in the indexing of the given sum to simplify its general term.

54. $\sum_{i=5}^{25} 2(i-4)^3$

55. $\sum_{j=3}^{n} \frac{1}{j-2}$

56. $\sum_{k=0}^{n} n(k-3)^2$

57. $\sum_{l=4}^{20} \cos((2l+2)\pi)$

58–63 Follow the lead of Examples 2 and 4 in using the limit process to find the area under the graph of $f(x)$ and above the x–axis on the given interval. (In Exercises 58 and 59, use a formula from geometry to check your answer.)

58. $f(x)=\frac{1}{2}x+1$ on $[0,4]$

59. $f(x)=5-x$ on $[1,3]$

60. $f(x)=x^2$ on $[1,2]$

61. $f(x)=x-x^3$ on $[0,1]$

62. $f(x)=x^2+3x$ on $[-2,2]$

63. $f(x)=(1-x^2)(1+x)$ on $[0,1]$

64–67 Identify the region whose area is the given limit. Do not evaluate the limit. (**Hint:** For guidance, see Example 4 and Exercises 58–63.)

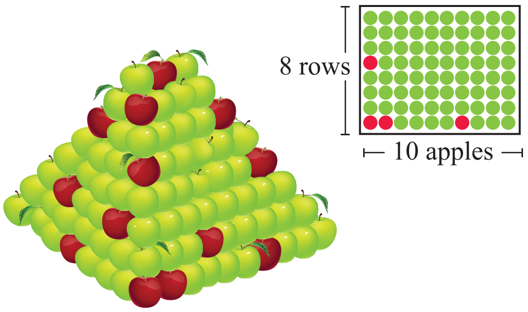
64. $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{i}{n}\right)^2$

65. $\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3+\frac{6i}{n}\right]$

66. $\lim_{n\to\infty} \frac{2}{n}\sum_{i=0}^{n-1}\sqrt{1+\frac{2i}{n}}$

67. $\lim_{n\to\infty} \frac{\pi}{2n}\sum_{i=0}^{n-1}\sin\frac{\pi i}{2n}$

68. A fruit vendor stacks apples in a rectangular, pyramid-like pile. If the foundational layer consists of 8 rows of 10 apples, and the top layer is a single row of apples, find how many apples are in the stack. Generalize to the case of an $m \times n$ bottom layer of fruit.



69. In statistics, the standard deviation of a data set $x_1, x_2, ..., x_n$ is defined to be the square root of the average of the squares of deviations of the data from their mean $\bar{x}$.

$$s=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n}}$$

Rewrite the definition of s using sigma notation, and use summation facts to derive the following formula for the variance of the data set, which is the square of the standard deviation.

$$s^2=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-\bar{x}^2$$

70. Find the distance covered by the pebble in Exercise 79 of Section 4.7 from $t=2$ seconds to $t=5$ seconds. (**Hint:** Find the velocity function first.)

71. Assuming constant acceleration, use the method of Exercise 70 to find the distance covered by the Bugatti in Exercise 91 of Section 4.7 from $t=1$ second to $t=3$ seconds.

72. Assuming constant deceleration, use the method of Example 2 to find the distance covered by the braking race car in Exercise 90 of Section 4.7 from $t=1$ second to $t=2$ seconds. (See the hint given in Exercise 70.)

73. The velocity function of a moving object is given by $v(t)=9-0.5t^2$ m/s from $t=0$ s to $t=3$ s. Find the distance covered by the object during this time.

74. Repeat Exercise 73 for $v(t)=4-0.5t^3$ on the interval $[0,2]$.

75. Use geometry to show that the shaded area under the curve $s(t)$ in Example 2 is 128 square units. (**Hint:** Divide the region into a rectangle and a right triangle, and use well-known area formulas. Alternatively, you may want to use the area formula for a trapezoid.)

76. Show that you can obtain the same answer in Example 2 by evaluating $\lim_{n\to\infty} U_n$, that is, by choosing $t_i^* = t_{i-1}$ for every index value i.

77. Show that you can obtain the same answer in Example 4 by evaluating $\lim_{n\to\infty} O_n$, that is, by choosing $x_i^* = x_i$ for every index value i.

78. Show that you can obtain the same answer in Example 2 by choosing t_i^* to be the midpoint of the i^{th} interval for every index value i.

79. Use an elementary argument to prove the following summation formula.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(**Hint:** Letting $S = \sum_{i=1}^{n} i$, add to S its terms in "reverse order"; that is, calculate $2S$ as $2S = \sum_{i=1}^{n} i + \sum_{j=0}^{n-1} (n-j)$, and notice that, after rearranging terms, this latter sum equals $(1+n)+(2+(n-1))+(3+(n-2))+\cdots$ $=(n+1)+(n+1)+(n+1)+\cdots$.

Use this observation to complete the argument. Note that this argument is attributed to C. F. Gauss, who discovered it as a barely nine-year-old elementary school student.)

80.* Use mathematical induction to establish the summation formula of Exercise 79.

81.* Use mathematical induction to establish the following summation formula.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

82.* Use mathematical induction to establish the following summation formula.

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

83.* Prove the summation formula of Exercise 82 by making use of the following identity.

$$(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$$

84.* Inscribe a regular n-gon in a circle of radius r. Use radii to divide the n-gon into n isosceles triangles, and add the areas of the triangles to obtain the area of the inscribed n-gon. Finally, let $n \to \infty$ to obtain the area formula for the circle.

85–87 *Double summations* are important in many areas of mathematics, statistics, computer science, and the sciences in general. They have the form $\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}$.

Evaluate the given double sum.

85. $\sum_{i=1}^{4}\sum_{j=1}^{5} (i+j)$

86. $\sum_{i=1}^{5}\sum_{j=1}^{6} ij$

87. $\sum_{i=1}^{n}\sum_{j=1}^{m} ij$

5.1 Technology Exercises

88–91 Use a graphing utility to express the area under the graph of $f(x)$ and above the x-axis on the indicated interval as a limit. Then use technology to evaluate the limit to find the area.

88. $f(x) = x^6$ on $[0,1]$

89. $f(x) = \sin x$ on $[0,\pi]$

90. $f(x) = e^x$ on $[1,2]$

91. $f(x) = x + \cos^2(\pi x)$ on $\left[0,\frac{1}{2}\right]$