Solution

The given information consists only of the two angles and the two speeds, but in the figure we have already labeled other quantities that might help us relate the angles and speeds and arrive at Snell's Law. Specifically, we have let p denote the horizontal distance between the object B and the point directly beneath the observer at A. And since the point C of refraction (where the rays of light bend) is unknown, we have given its horizontal displacement from A the label x, meaning the horizontal distance between C and B is $p - x$. The vertical distances a and b are fixed, but the lengths of the hypotenuses, d_1 and d_2, will vary as x varies.

At this point, it may very well be unclear how to arrive at Snell's Law from what we have. But we haven't yet applied Fermat's Principle, and we can deduce many relationships between the labeled quantities. To begin with, since distance = rate · time, the time it takes light to travel from B to C is d_2/v_2 and the time it takes to travel from C to A is d_1/v_1. So the total time is expressed as follows.

$$T = \frac{d_1}{v_1} + \frac{d_2}{v_2}$$

We can express the two hypotenuses as functions of x by noting that $a^2 + x^2 = d_1^{\,2}$ and $b^2 + (p-x)^2 = d_2^{\,2}$, so

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (p-x)^2}}{v_2}$$

and the domain of T is $[0, p]$.

The actual distance x must be the value of x that minimizes T, so our next step is to find T'.

$$T'(x) = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{p-x}{v_2\sqrt{b^2 + (p-x)^2}}$$

$$= \frac{x}{v_1 d_1} - \frac{p-x}{v_2 d_2} \qquad \text{Substitute } d_1 \text{ and } d_2 \text{ for their formulas.}$$

$$= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \qquad \sin\theta_1 = \frac{x}{d_1} \text{ and } \sin\theta_2 = \frac{p-x}{d_2}$$

Note that $T_+'(0) < 0$ and $T_-'(p) > 0$, so by Darboux's Theorem (Section 3.1), there is a point $x \in [0, p]$ for which $T'(x) = 0$. And by the First Derivative Test, that point must minimize T. Rewriting $T'(x) = 0$, we have developed the formula for Snell's Law.

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \quad \text{or} \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

4.6 Exercises

1. Find two integers whose sum is 120 and whose product is as large as possible. (**Hint:** If you denote the first number by x, then the second number is $120 - x$. Now write a formula for the product, and use calculus to find the maximum.)

2–14 Use the strategy suggested in Exercise 1 to find two numbers satisfying the given requirements.

2. The sum is S and the product is a maximum.

3. The difference is 36 and the product is as small as possible.

4. Two positive numbers whose product is 144 and the sum is a minimum.

5. Two positive numbers whose product is n^2 and the sum is a minimum.

6. Two positive numbers whose product is 162 and the sum of twice the first and the second is a minimum.

7. Two positive numbers that are reciprocals of each other and their sum is a minimum.

8. Two positive integers so that the square of the first number plus the second number is 243 and their product is a maximum.

9. The sum of twice the first and three times the second is 480 and their product is a maximum.

10. The product of two positive integers is 32 and the sum of twice the first plus the second is a minimum.

11. Two positive numbers whose product is 16 and the sum of whose squares is a minimum.

12. Two positive numbers whose sum is 1 and the sum of whose cubes is a minimum.

13. Two nonnegative numbers whose sum is 1 and the sum of whose cubes is a maximum.

14. Repeat Exercises 12 and 13 using fourth powers instead of cubes.

15. Modify Example 2 by inscribing a rectangle in the region bounded by the x-axis and the parabola $y = k - x^2$ $(k > 0)$.

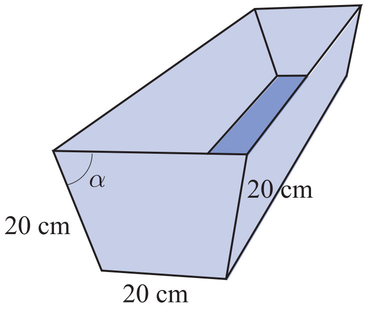
16. A vertex of a rectangle is at the origin; the opposite vertex sits in the first quadrant and on the line $2y + x = 4$. Find the dimensions that maximize the area of such a rectangle.

17. Repeat Exercise 16 with the opposite vertex sitting on the graph of $y = 32 - x^3$.

18. From among all lines through the point $(3,1)$, find the one forming with the coordinate axes a right triangle of minimum area.

19. Repeat Exercise 18, this time finding the line forming a triangle whose hypotenuse is of minimum length.

20. Suppose that when constructing a trough similar to the one in Exercise 35 of Section 3.8, both the shorter base and the legs of its cross-section are 20 centimeters long. Find the base angle α that maximizes the volume of the trough.



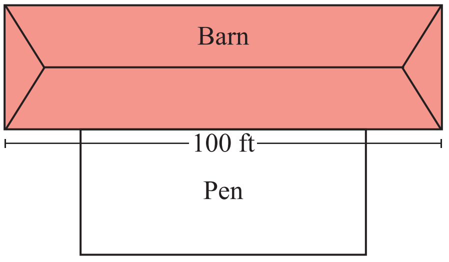
21. Find the coordinates of the point on the graph of $y = \sqrt{x}$ that is closest to the point $(1,0)$.

22. Find the coordinates of the point on the graph of $y = x^3$ that is closest to the point $(-4,0)$.

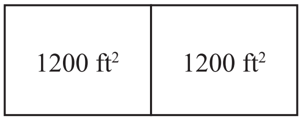
23. Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $2x^2 + 6y^2 = 12$.

24. Find the equation of the line tangent to the graph of $y = 1 - x^2$ that forms with the coordinate axes the triangle of minimum area in the first quadrant.

25. A farmer has 120 feet of fencing to construct a rectangular pen up against the straight side of a barn, using the barn for one side of the pen. The length of the barn is 100 feet. Determine the dimensions of the rectangle of maximum area that can be enclosed under these conditions. (**Hint:** Be mindful of the domain of the function you are maximizing.)

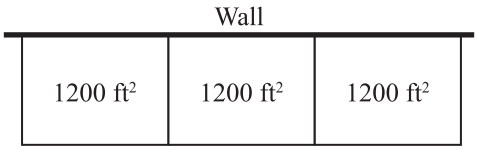


26. A farmer needs to construct two adjoining rectangular pens of identical areas, as shown. If each pen is to have an area of 1200 square feet, what dimensions will minimize the cost of fencing?

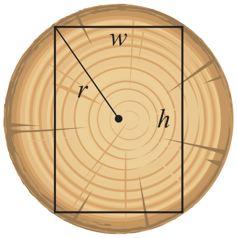


27. Repeat Exercise 26 if the pens are constructed against a straight wall that serves as a side for each.

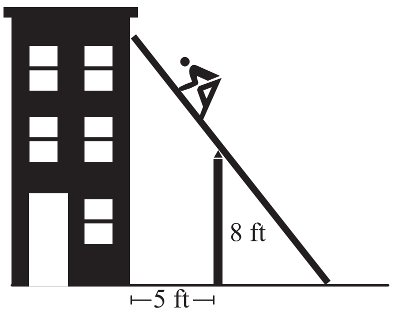
28. Repeat Exercise 27 if three identical adjoining pens are to be constructed, as shown.



29. A supporting beam with a rectangular cross-section is to be cut from a log that has an approximately circular cross-section with a radius of r inches. Knowing that the strength of such a beam is directly proportional to the width multiplied by the second power of the height of its cross-section, find the strongest beam that can be cut under these conditions.



30. An 8-foot fence stands 5 feet from a tall building. A contractor needs to reach the building with a ladder from the outside of the fence. Find the minimum length of the ladder that can do the job.



31. A 30 in. piece of wire is cut and the pieces are bent into a circle and a square, respectively. Where should we cut in order to minimize the sum of the areas of these two shapes?

32. Repeat Exercise 31, this time producing an equilateral triangle and a square.

33. Repeat Exercises 31 and 32, this time maximizing the sum of the two areas.

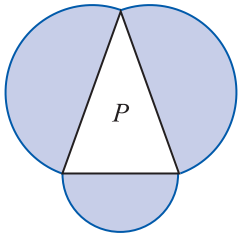
34. Prove that among all rectangles that can be inscribed in a circle, the square has the greatest perimeter.

35. Prove that among all isosceles triangles of a given area, the equilateral triangle has the minimum perimeter.

36. Find the dimensions of the rectangle whose perimeter is P units and area is a maximum.

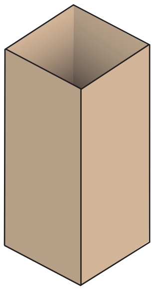
37. Find the dimensions of the rectangle whose area is A units and perimeter is a minimum.

38. The perimeter of an isosceles triangle is P inches. Find the side lengths so as to minimize the sum of areas of the semicircles drawn onto the sides of the triangle.



39. Suppose we want to construct a can in the shape of a right circular cylinder with no top whose surface area is to be S square inches. What dimensions will maximize the volume?

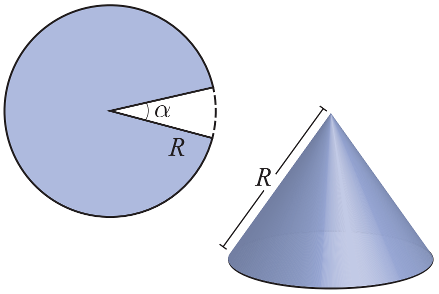
40. If we want to make a rectangular box with a square bottom and no top that holds 32 cubic inches, and the construction material costs 3 cents per square inch, what are the dimensions and the cost of the least expensive box that can be made?



41. If the box to be constructed in Exercise 40 is to hold the same volume, but we need to construct a top from an expensive, heat-resistant material that costs 21 cents per square inch, how does the new requirement change the cost and dimensions of the least expensive box?

42. Determine the dimensions and maximum volume of the rectangular box with no top and a square base if its surface area is A square inches.

43. A cone is to be constructed by cutting out a sector of central angle α of a disk of radius R and gluing the cut lines together to form a cone. Find the value of α that maximizes the volume of the cone.

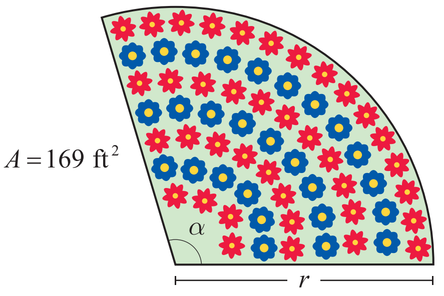


44. The pages of a children's book are to contain 54 square inches of printed matter and illustrations, with margins of 1 inch along the sides and $1\frac{1}{2}$ inches along the top and bottom of each page. Find the dimensions of the page that will require the minimum amount of paper.

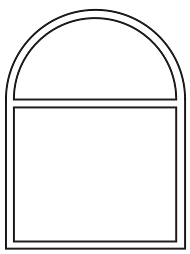
45. A poster is to contain 150 square inches of printed matter, surrounded by margins that are 3 inches wide on the top and bottom, and 2 inches on each side. Find the dimensions for the poster that minimize its total area.

46. The sum of squares of lengths of the sides of a right triangle is 64 square inches. Find the side lengths that maximize the area of the triangle.

47. A flower bed is planned in the form of a circular sector. Find the central angle and radius if it is to cover 169 square feet, and its perimeter is to be a minimum.



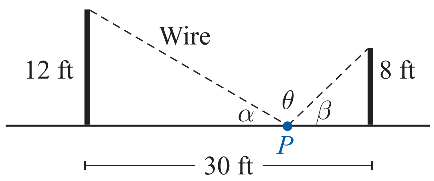
48. The shape of a Norman window can be approximated by a rectangle with a semicircle on top. What dimensions will admit the maximum amount of light if the perimeter of the window is to be P inches?



49. In Exercise 103 of Section 3.6, find the optimum distance s that maximizes the viewing angle.

50. An office building is located right on a riverbank, which is straight. A small power plant is on the opposite bank, 1500 feet downstream from the point directly opposite the office building. The river is 300 feet wide. If we want to connect the power plant and the building by cable, which costs \$1700 per foot to lay down underwater and \$800 per foot underground, what is the least expensive path for the cable?

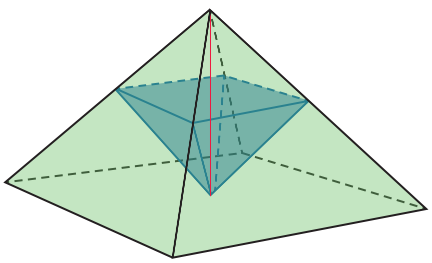
51. Two antennas standing 30 feet apart are to be stayed with a single wire. The wire runs from the top of the first antenna, is secured to the ground somewhere between the antennas, and is finally attached to the top of the second antenna. If the height of the first antenna is 12 feet, while that of the second is 8 feet, find the point along the line segment connecting the bases where the wire needs to be staked to the ground if the length of the wire is to be minimal.



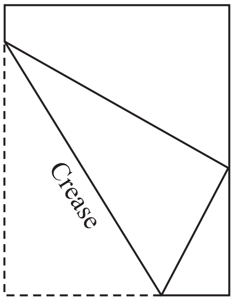
52. If we denote the heights of the antennas in Exercise 51 by h_1 and h_2, respectively, and the distance between them is d, prove that the wire has minimal length if and only if $\alpha = \beta$.

53. In Exercise 51, find the location of P that maximizes the angle θ.

54.* An inverted square pyramid is to be inscribed into a larger square pyramid of volume V, so that the two have a common axis, and the vertex of the inscribed pyramid coincides with the center of the outer pyramid's base. Find the ratio of the pyramids' altitudes so that the volume of the inscribed pyramid is maximal.



55.* The lower left corner of a letter-sized paper, which is 8.5 in. by 11 in., is folded over to reach the right edge of the paper. Find a way that this can be done so as to produce a crease of minimum length.



56. Find the radius of the base and the height of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

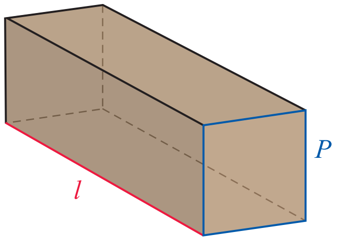
57. Repeat Exercise 56, but inscribe a right circular cone instead of a cylinder in the sphere of radius R.

58. Find the radius of the base and the height of the right circular cylinder of largest volume that can be inscribed in a circular cone if the height of the cone is H and its base has radius R.

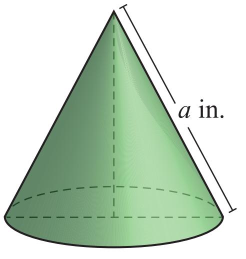
59. Repeat Exercise 58, but find the extremum of the surface area of the cylinder instead of its volume.

60. The sum of the height and the radius of the base of a circular cylinder is 12 inches. Find their lengths if the volume of the cylinder is to be a maximum.

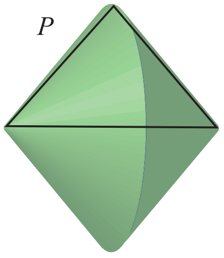
61. Suppose that we want to send a parcel in the shape of a square-based rectangular solid, and the Standard Post service limits the sum of the length and girth (girth = the perimeter of the base) to 130 inches. Find the dimensions of the package of the greatest volume under these conditions.



62. Find the maximum volume a right circular cone can have if its slant height is a inches.



63. An isosceles triangle of perimeter P is rotated around its base. What base length will produce the solid of maximum volume?



64. A lighthouse is 2 miles off a straight shoreline, and a grocery store is 10 miles down the coast. If the lighthouse keeper can row at 2.4 mph and walk at 4 mph, where should he land in order to make the best time to the store to get supplies? What if he is picked up by a golf cart that can drive at 9.9 mph?

65. Repeat Exercise 64 if the lighthouse keeper uses a motorboat whose top speed is 20.1 mph, and will be picked up by a car that will drive at the posted speed limit of 45 mph.

66. Repeat Exercise 64 if the lighthouse and the store are both on the shore of a circular lake of diameter d at the endpoints of the diameter.

67. At noon on a certain day, a plane is 200 miles south of another airliner and flying north at 550 mph, while the second plane is flying southwest at 600 mph. How much later after this instant is their distance a minimum?

68. A straight two-lane highway intersects a straight interstate at a right angle. A car exits the interstate and starts moving away from it on the two-lane highway at 50 mph. At the same instant, another car, moving at 75 mph on the interstate, is approaching the same intersection, but is still 10 miles from it. When will their distance be a minimum and what will this distance be?

69. The position of an object connected to a spring is given by $d(t) = \sin 3t + \cos 3t$, where d is measured in feet, and t in seconds. Find when the absolute value of its velocity first reaches its maximum and the value of the maximum velocity.

70. In Exercise 69, find when the absolute value of the acceleration first reaches its maximum and the value of this acceleration.

71. Ignoring air resistance, the range r of a projectile fired from the ground in a flat area with an initial velocity of v_0 can be calculated by $r = \left(v_0^2/g\right)\sin 2\theta,$ where g is the gravitational acceleration and θ is the launch angle relative to the horizontal. Find the launch angle that maximizes the range if the initial velocity is a given constant.

72. The luminance E_l at distance d from a light source is directly proportional to the light intensity F_l (also called luminous flux) and inversely proportional to the square of distance: $E_l = F_l/\left(4\pi d^2\right).$ Suppose two light bulbs are 3 meters apart, with respective light intensities of $F_{l,1} = 1700$ lumens (lm) and $F_{l,2} = 1000$ lm. Where between these light bulbs will the sum of their luminance levels be a minimum?

73. Management and Power, Inc. has found that its seminar on management techniques attracts 800 people when the seminar fee is set to \$600. They estimate that for each \$15 discount in the charge, an additional 50 people will attend the seminar. Find the amount that Management and Power, Inc. should charge for the seminar to maximize the revenue, and find the maximum revenue.

74. A blueberry farmer owns 1056 plants, each producing p pounds on average during a regular season. He estimates that for each additional dozen of new plants planted on his farm, average production per plant is going to drop by a half percent. What would be the optimum number of plants on the farm in order to maximize production, and what is the optimum production level?

75. The manager of a 115-unit apartment complex finds that all units are rented at a price of \$1500 per month. Research shows that for each \$20 increase in rent, one additional unit remains vacant. How much should he charge for rent in order to bring in maximum revenue, and how many units are rented then?

76. A moving company sends a truck on a 2000-mile round-trip to move two households. The hourly fuel consumption of the truck is approximated by $2 + \frac{1}{280.1}v^2$ gallons, where v is assumed to be a constant speed somewhere between 35 and 70 miles per hour. If a gallon of diesel fuel costs \$4.50 and the driver is paid \$35 an hour, what speed will minimize the company's transportation costs?

77. Cool Wheels, a manufacturer of die-cast model cars, has a monthly overhead cost of \$6000, material costs of \$2 per toy car, and each has associated labor costs of \$0.40. When producing and marketing 2500 cars a month, each sells for \$30.75. When producing more, it was found that for each additional 100 units, the market conditions cause the price to drop by a dollar. In addition, labor costs go up by 5 cents for each additional 100 units because of expensive overtime pay. Find the production level and selling price that maximize the profit under these conditions.

78. Suppose it costs a candy company \$3 to produce and distribute a box of Chi-Can chipotle candy bars, and the number of boxes sold at x dollars a box is approximated by $n = \dfrac{80}{x-11} + 15(50 - x).$ What sale price will bring the maximum profit?

79. Prove that when the company in Exercise 78 maximizes its profit, the marginal cost equals the marginal revenue.

80. Suppose that $R(x) = 2x^3 - 15x^2$ and $C(x) = 3x^3 - 25x^2 + 21x$ are the weekly revenue and cost functions for a particular commodity, where x represents units of 100 individual products and where the model is thought to be accurate up to approximately $x = 10$. What is the profit zone, and what level of production will maximize the profit? (See Example 4.)

81. The cost of manufacturing x units of a commodity is given by $C(x) = x^3 - 15x^2 + 12{,}000x.$ Find the value of x that minimizes the average cost of production. (See Example 5.)

4.6 Technology Exercises

82–83 Use the graphing and symbolic differentiation capabilities of a computer algebra system to solve the problem.

82. Suppose we have a small supply of craft paint, enough for 1 square foot, and we want to use it to paint a regular tetrahedron and a cube from a children's toy set. What should the dimensions of these solids be if we want to maximize the total volume? How about minimizing the volume?

83. Repeat Exercise 82 for a tetrahedron and a sphere.