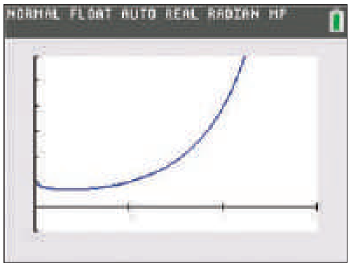

Figure 3

$y = x^x$ on $[0,3]$ by $[-1,6]$

$$\ln y = x \ln x = \frac{\ln x}{\frac{1}{x}},$$

which gives us a limit of indeterminate form ∞/∞. So,

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0,$$

and hence $x^x \to e^0 = 1$ as $x \to 0^+$ (don't forget this last step!).

Example 9 Using L'Hôpital's Rule to Find a Limit of Indeterminate Form ∞^0

Determine $\lim_{x\to\infty} x^{1/x}$.

Solution

The base has a limit of ∞ and the exponent has a limit of 0. We proceed as in the last two examples.

$$y = x^{1/x}$$

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x} \qquad \text{Indeterminate form } \infty/\infty$$

Applying L'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

and therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} y = e^0 = 1$.

4.4 Exercises

1–12 Evaluate the limit using the theorems of Chapter 2. Then decide whether L'Hôpital's Rule is applicable and, if so, use it to check your answer.

1. $\lim_{x\to 3} \frac{2x^2 - 18}{x-3}$

2. $\lim_{x\to -2} \frac{x^3 + 8}{x+2}$

3. $\lim_{x\to 0} \frac{\sin x}{2x}$

4. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

5. $\lim_{x\to 0} \frac{\cos x}{x}$

6. $\lim_{x\to\infty} \frac{6x^2 - x + 7}{x - 3x^2}$

7. $\lim_{x\to -\infty} \frac{5x^2 - 2x + 1}{2.5x^3 - 3x^2 + 6}$

8. $\lim_{x\to 0} \frac{2x}{\sqrt{x+3} - \sqrt{3}}$

9. $\lim_{x\to 0} \frac{\sec x}{x}$

10. $\lim_{x\to 0^+} \left(\sqrt{x}\right)^{1/x}$

11. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x\sqrt{x+1}}\right)$

12. $\lim_{x\to 0} \frac{x}{3\tan x}$

13–16 Two functions are in competition to determine the indicated limit. Identify the type of the indeterminate form, and fill out the table to decide which function dominates.

13. $\lim_{x\to\infty} f(x)$, where $f(x)=\dfrac{\sqrt{5x^3+7}}{0.2x^2+1}$

x	1	10	100	1000	10,000	100,000
$f(x)$						

14. $\lim_{x\to\infty} g(x)$, where $g(x)=\dfrac{0.5\sqrt{x}}{\ln(x+1)}$

x	1	10	100	1000	10,000	100,000
$g(x)$						

15. $\lim_{x\to\infty} h(x)$, where $h(x)=x^{100}e^{-x}$

x	1	10	100	1000	10,000	100,000
$h(x)$						

16. $\lim_{x\to 0} k(x)$, where $k(x)=(\sin x)^x$

x	1	0.5	0.1	0.01	0.001	0.0001
$k(x)$						

17–48 Check whether L'Hôpital's Rule applies to the given limit. If it does, use it to determine the value of the limit. If it does not, find the limit some other way. (When necessary, apply L'Hôpital's Rule several times.)

17. $\lim_{x\to\infty}\dfrac{2x+5}{x^2-7}$

18. $\lim_{x\to\infty}\dfrac{4-2.5x}{x+3}$

19. $\lim_{x\to-\infty}\dfrac{1.5x^3-2x^2+x+9}{x^2+2.1x-4}$

20. $\lim_{x\to-\infty}\dfrac{4.5x^4+x^3-2}{3-1.5x^4}$

21. $\lim_{x\to 0^+}\dfrac{\sqrt{x}}{\ln x}$

22. $\lim_{x\to\infty}\dfrac{x\sin x}{e^{-x}}$

23. $\lim_{x\to\infty}\dfrac{\frac{1}{x}+2}{2x+1}$

24. $\lim_{t\to 0}\dfrac{t}{\sqrt{2t+9}-3}$

25. $\lim_{x\to\infty}\dfrac{\sin x+2\ln x}{x^2+5}$

26. $\lim_{x\to 0}\dfrac{\sin x-x}{1-\cos x}$

27. $\lim_{t\to 0}\dfrac{1-\cos t}{3t}$

28. $\lim_{x\to -1^+}\dfrac{\sin\sqrt{x+1}}{x+1}$

29. $\lim_{x\to 0^+}\dfrac{x^{3/2}}{\ln(\cos x)}$

30. $\lim_{x\to 0}\dfrac{\ln(\sec^2 x)}{\sqrt{x}}$

31. $\lim_{x\to\infty}\dfrac{\ln x}{\ln(x^2+3x)}$

32. $\lim_{x\to 0}\dfrac{\log_{10}(x^2+2x+1)}{\log_{10}(x+1)}$

33. $\lim_{x\to 0}\dfrac{x}{3^{x/2}-1}$

34. $\lim_{x\to\infty}\dfrac{2^x}{x^2-3x+4}$

35. $\lim_{x\to 0}\dfrac{\sin x-x}{3x^2}$

36. $\lim_{\phi\to 0^+}\dfrac{1-\cos\phi}{\csc\phi}$

37. $\lim_{\alpha\to 0}\dfrac{\alpha}{e^{\sin\alpha}-1}$

38. $\lim_{\theta\to 0}\dfrac{\theta\tan\theta}{1-\cos\theta}$

39. $\lim_{t\to\infty}\dfrac{\ln(t+1)}{e^{-t}\sin t}$

40. $\lim_{t\to\pi}\dfrac{(\cos(2t)-1)^2}{t-\pi}$

41. $\lim_{\theta\to\pi/2}\dfrac{\left(\theta-\frac{\pi}{2}\right)^2}{\ln(\sin\theta)}$

42. $\lim_{x\to\infty}\dfrac{x+2^x}{5^x-x}$

43. $\lim_{x\to\infty}\dfrac{4^x+x^2}{3^x-x}$

44. $\lim_{x\to\infty}\dfrac{\ln(\ln x)}{x\ln x}$

45. $\lim_{x\to 0^+}\dfrac{\log_2(1+x)}{\log_3(\sin x+1)}$

46. $\lim_{x\to\infty}\dfrac{\log_4(2x+1)}{\log_5(x-4)}$

47. $\lim_{x\to 0^+}\dfrac{\log_4(x+1)}{\log_3 x}$

48. $\lim_{x\to 0}\dfrac{3^x-1}{x3^x}$

49–74 Identify the indeterminate product, quotient, difference, or power, and use L'Hôpital's Rule to find the limit. If the limit is not of indeterminate form, say so and find it by other means.

49. $\lim_{x\to 0^+} x\ln x$

50. $\lim_{x\to\infty}\dfrac{\sqrt{2x^2+1}}{x+3}$

51. $\lim_{x\to 0} x\cos\frac{\pi}{x}$

52. $\lim_{x\to\infty}(\ln x)^{-1/x}$

53. $\lim_{x\to 0^+}\left(\frac{1}{x}\right)^x$

54. $\lim_{x\to 0^+}(-\ln x)^x$

55. $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{2}{x-1}\right)$

56. $\lim_{x\to 0^+}\left(\frac{1}{x}+\ln x\right)$

57. $\lim_{x\to 4^+}\left(\frac{32}{x^2-16}-\frac{x}{x-4}\right)$

58. $\lim_{x\to 0^+} x^{\left(x^2\right)}$

59. $\lim_{x\to 0^+}\left(2^x-x\right)^{1/x}$

60. $\lim_{x\to 0^+}(1-x)^{1/x}$

61. $\lim_{x\to 0^+}\left(\frac{1}{x^2}\right)^{\csc x}$

62. $\lim_{x\to\infty}\left(\sqrt{x^2-3x}-\frac{3}{x^2+1}\right)$

63. $\lim_{x\to\infty}(x-1)^{1/x}$

64. $\lim_{x\to\infty}\frac{\ln x}{x^{7/5}}$

65. $\lim_{x\to 0}(\cos x)^{\cot x}$

66. $\lim_{x\to 0^-}(\cot x)^{\cos x}$

67. $\lim_{x\to 0^+}\tan x\sec x$

68. $\lim_{x\to\infty}\frac{x^{100}}{3^x}$

69. $\lim_{x\to\infty}\frac{\ln\left(100x^2+e^x\right)}{100x}$

70. $\lim_{x\to 0^+} 2\sqrt{x}\csc x$

71. $\lim_{x\to 0}(1+2x)^{1/x}$

72. $\lim_{x\to(\pi/2)^-}\left(\frac{\pi^2}{4}-x^2\right)\sec x$

73. $\lim_{x\to 0}\frac{\sin 2x}{\tan 3x}$

74. $\lim_{x\to 1} x^{1/(1-x)}$

75–85 Find the limit. If applicable, use L'Hôpital's Rule (as many times as appropriate).

75. $\lim_{x\to\infty}\frac{2x^5+x^3-4}{e^x}$

76. $\lim_{x\to\infty}\frac{\cos x}{2^x}$

77. $\lim_{x\to\infty} x\sin\frac{1}{x}$

78. $\lim_{x\to\infty} x^{1/x^3}$

79. $\lim_{x\to 0^+} x^{x^x}$

80. $\lim_{x\to 0^+}\left(x^x\right)^x$

81. $\lim_{x\to\infty} x^{1/x^n},\quad n\in\mathbb{Z}^+$

82. $\lim_{x\to\infty}\frac{(\ln x)^3}{x^2}$

83. $\lim_{x\to 0}\frac{\sin x-x}{2x-e^x+e^{-x}}$

84. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{x^2}$

85. $\lim_{x\to 0}\frac{\sin x-x}{\tan x-x}$

86–91 Find the error(s) in the limit calculation.

86. $\lim_{x\to 0}\frac{1-\sin x}{x}=\lim_{x\to 0}\frac{-\cos x}{1}=-1$ (Incorrect!)

87. $\lim_{x\to 2}\frac{x^2-2}{x-2}=\lim_{x\to 2}\frac{2x}{1}=4$ (Incorrect!)

88. $\lim_{x\to-\infty}\frac{5^x+1}{5^x}=\lim_{x\to-\infty}\frac{(\ln 5)5^x}{(\ln 5)5^x}=1$ (Incorrect!)

89. $\lim_{x\to 0^+} x\cot x=\lim_{x\to 0^+}(1)\left(-\csc^2 x\right)$

$=(1)(-\infty)=-\infty$ (Incorrect!)

90. $\lim_{x\to 0} x\sin\frac{1}{x}=\lim_{x\to 0}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$

$=\lim_{x\to 0}\frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}}$

$=\lim_{x\to 0}\cos\frac{1}{x}$

= does not exist (Incorrect!)

91. $\lim_{x\to 0^-}\frac{\cos x-x^2-1}{x^4-2x^3}=\lim_{x\to 0^-}\frac{-\sin x-2x}{4x^3-6x^2}$

$=\lim_{x\to 0^-}\frac{-\cos x-2}{12x^2-12x}$

$=\lim_{x\to 0^-}\frac{\sin x}{24x-12}=0$ (Incorrect!)

92–106 Convince yourself that the initial use of L'Hôpital's Rule is not helpful in finding the limit. If possible, try to find a way to make use of the theorem, or evaluate the limit in some other way.

92. $\lim_{x\to\infty}\frac{\sqrt{x+2}}{\sqrt{x}}$

93. $\lim_{x\to\infty}\frac{\sqrt[3]{x+1}-2}{\sqrt{x^2+2}}$

94. $\lim_{x\to\infty}\frac{2^x+3^x}{5^x}$

95. $\lim_{x\to\infty}\frac{5^x-6^x}{7^x+8^x}$

96. $\lim_{x\to\infty}\frac{2^{-x}}{x^{-1}}$

97. $\lim_{x\to\infty}\left(\frac{1}{x+1}\right)^{-x^3}$

98. $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{e^{-x}}$

99. $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

100. $\lim_{x\to 0}\frac{\csc x}{\cot x}$

101. $\lim_{x\to 0^+}\left(\cot x-\frac{5x+1}{x}\right)$

102. $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$

103. $\lim_{x\to\pi^+}(\cot x)^{\sin x}$

104. $\lim_{x\to\infty} 2^{-x}x\ln x$

105. $\lim_{x\to 0}(\sin x)^{\tan x}$

106. $\lim_{x\to(\pi/2)^-}\left(\frac{1}{\frac{\pi}{2}-x}-\tan x\right)$

107–110 Find the limit of the sequence by considering the function you obtain after replacing *n* with the real variable *x*.

107. $\lim_{n\to\infty}\frac{n^2+1}{2^n}$

108. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

109. $\lim_{n\to\infty}\sqrt[n]{n}$

110. $\lim_{n\to\infty}\frac{2^n+5^n}{6^n}$

111–114 Use L'Hôpital's Rule to prove the assertion.

111. $\lim_{x\to 0}\frac{\sin(kx)}{x^k}=\infty \quad (k>1)$

112. $\lim_{x\to\infty}\frac{p(x)}{e^{kx}}=0 \quad (p(x) \text{ is a polynomial, } k>0)$

113. $\lim_{x\to\infty}\frac{(\ln x)^n}{x^k}=0 \quad (n\in\mathbb{N},\ k>0)$

114. $\lim_{x\to\infty}\frac{a^x}{x^n}=\infty \quad (a>1,\ n\in\mathbb{N})$

115–122 Find the value(s) of *c* satisfying the conclusion of Cauchy's Mean Value Theorem. If the theorem doesn't apply, explain why.

115. $f(x)=x, \quad g(x)=x^2+1; \quad [0,1]$

116. $f(x)=x^3-1, \quad g(x)=x^2+2x; \quad [-1,1]$

117. $f(x)=x^3-x, \quad g(x)=-x^2+2x+3; \quad [-1,3]$

118. $f(x)=x^3, \quad g(x)=-x^2; \quad [-2,3]$

119. $f(x)=x^2+3x, \quad g(x)=3x^2-5x+3; \quad [-1,3]$

120. $f(x)=\frac{1}{x}, \quad g(x)=\ln x; \quad [1,2]$

121. $f(x)=\cos x, \quad g(x)=\sin x; \quad \left[-\frac{\pi}{2},0\right]$

122. $f(x)=x^2-5x-9, \quad g(x)=x^3+x+10; \quad [-3,2]$

123–124 Prove that $f(x)$ has a removable discontinuity at $x=0$. Then find the value of *c* so as to make *f* continuous.

123. $f(x)=\begin{cases}\frac{3\tan x-2x}{5x^2+3x} & \text{if } x\neq 0\\ c & \text{if } x=0\end{cases}$

124. $f(x)=\begin{cases}(e^x-\sin 2x)^{2/x} & \text{if } x\neq 0\\ c & \text{if } x=0\end{cases}$

125. Recall the following compound interest formula for the value of an investment of *P* dollars after *t* years, compounded *n* times a year at an annual interest rate of *r*.

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$

Use L'Hôpital's Rule to prove that if we let $n\to\infty$, we obtain the following continuous compounding formula.

$$A=Pe^{rt}$$

126. The strength of an electric field due to a disk charge is obtained from the formula

$$E(x)=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$$

where σ is the electric charge per unit area (in $\mathrm{C/m^2}$), $\varepsilon_0=8.85\cdot 10^{-12}\ \mathrm{C^2/Nm^2}$, *R* is the radius of the ring, and *x* is the distance to the charge in meters. Use L'Hôpital's Rule to confirm that $E(x)\to 0$ as $x\to\infty$. How is *E* affected by σ and *R* at a given distance? What happens to the rate of change of *E* as *x* increases? (**Hint:** Apply L'Hôpital's Rule to dE/dx as $x\to\infty$.)

127. Marquis de l'Hôpital first illustrated the rule named after him in his 1696 textbook, *Analyse des Infiniment Petits*. He used an example where the objective was to find

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$$

for $a>0$. Determine the above limit.

4.4 Technology Exercises

128–131 Check whether the limit is of indeterminate form, and then use a graphing utility to evaluate the limit.

128. $\lim_{x\to 1^+}(x-1)^{\ln x}$

129. $\lim_{x\to 0^+}\tan x\ln x$

130. $\lim_{x\to 0^+}x^{x^{x^x}}$

131. $\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$

132–133 Use a graphing utility to graph the function for different values of the parameter *c*. Examine how the values of the parameter affect the indicated limit.

132. $\lim_{x\to\infty}\left(1+\frac{1}{cx}\right)^x$

What happens to the limit when $|c|\to\infty$?

133. $\lim_{x\to 0^+}\frac{1-c^x}{cx}$

What happens to the limit when $c\to\infty$?