3.7 Exercises

1. Work through Example 1 with the following version of the growth model: $P(t) = P_0 a^t$, where a is treated as the (initially unknown) "growth constant." (**Hint:** Since P doubles every hour, $P(1) = 2P(0)$ gives $P_0 \cdot a^1 = 2P_0$.)

2. In an effort to control vegetation overgrowth, 100 rabbits are released in an isolated area that is free of predators. After one year, it is estimated that the rabbit population has increased to 500. Assuming exponential population growth, what will the population be after another six months?

3. The population of a certain inner-city area is estimated to be declining according to the model $P(t) = 237{,}000e^{-0.018t}$, where t is the number of years from the present. What does this model predict the population will be in 10 years?

4. A population of squirrels is growing in a Louisiana forest with a monthly growth constant of 6 percent. If the initial count is 100 squirrels, how many are there in a year? (**Hint:** Let $N(t)$ stand for the number of squirrels after t months, and note that $N'(t) = 0.06N(t)$. Mimic the steps of Example 1 or, alternatively, make use of the fact that $\frac{d}{dt}(a^t) = (\ln a)a^t$.)

5. The process of radioactive decay is akin to population growth in the sense that the rate of decay is proportional to the amount of material present at any given time. Therefore, it shouldn't come as a surprise that this process can be modeled with the same type of function. Suppose that $A(t)$ stands for the amount of a certain radioactive material at time t, and that it is decaying in a way that the rate of decay satisfies $\frac{d}{dt}A(t) = -0.1A(t)$ (note the negative sign), where t is measured in days. If we start with 1000 g of material, how much is left after 10 days?

6. In Exercise 67 of Section 1.2, we defined the *half-life* of a radioactive substance to be the amount of time required for half of the substance to decay. Find the half-life of the material in Exercise 5.

7. Carbon-11 has a radioactive half-life of approximately 20 minutes, and is useful as a diagnostic tool in certain medical applications. Because of the relatively short half-life, time is a crucial factor when conducting experiments with this element.
   a. Determine a so that $A(t) = A_0 a^t$ describes the amount of carbon-11 left after t minutes (as usual, A_0 is the amount at time $t = 0$).
   b. How much of a 2 kg sample of carbon-11 would be left after 30 minutes?
   c. How much of a 2 kg sample of carbon-11 would be left after 6 hours?

8. The half-life of radium-226 is approximately 4 days. Determine what percentage of the initial amount is left after two weeks.

9. According to Newton's Law of Cooling, the rate of change of temperature of a cooling object is proportional to the temperature difference between the object and the surrounding medium, that is,

$$\frac{dT(t)}{dt} = k\left[T(t) - T_s\right],$$

   where $T(t)$ is the temperature of the object at time t, and T_s is the temperature of its surroundings. Suppose that a cup of 180 °F coffee is left in a 72 °F room and cools to 122 °F in five minutes. How long does it take for the coffee to cool down to 85 °F? (**Hint:** Introduce a new variable for the temperature difference, $D(t) = T(t) - 72$, and then observe that Newton's law translates into the equation $D'(t) = k \cdot D(t)$. Now mimic the procedure seen in Example 1.)

10.* According to the Stefan-Boltzmann Law, the radiation energy emitted by a hot object of temperature T is $R(T) = kT^4$, where T is measured in kelvins. Use this to find a formula for the rate of change of energy emitted by the coffee of Exercise 9. (**Hint:** Use the following formula to convert degrees Fahrenheit to kelvins: $K = \frac{5}{9}(F - 32) + 273$.)

11. A snowplow is moving at a constant speed of 3 m/s, and the snow it is pushing is accumulating at a rate of 100 kg/s. What extra force is necessary for the engine of the snowplow to maintain constant speed despite the increasing mass?

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12. When washing his car, Brad is aiming a water hose at the side of the car, with water leaving the hose at a rate of 1 liter per second, with a speed of $15\ \text{m/s}$. If we ignore any "splash backs," what force does the water exert on the side of the car? (**Hint:** Use the equation $F = dP/dt$. See Example 2.)

13. Use Example 3 to find a formula for the force (in newtons) exerted on a scale by a rope dropped on it if the rope is 2 meters long and each centimeter of it has a mass of 20 grams. (As in the text, let x stand for the length of the segment of the rope that has already landed on the scale.)

14. Referring back to Exercise 12, suppose that after washing the car, with almost half of the contents of his 20-liter bucket still left, Brad pours it with a quick move into a much smaller 5-liter container that is sitting on the ground. Find the force exerted on the bottom of the smaller container by the incoming water at the instant when it starts overflowing. (**Hint:** Modify and use the result of Example 3.)

15. Suppose that the snowplow of Exercise 11 is 3500 kg and starts accelerating from $3\ \text{m/s}$ at a rate of $0.1\ \text{m/s}^2$. Assuming there is no snow accumulation this time, find the force exerted by the engine.

16.* Consider the accelerating snowplow of Exercise 15, this time assuming the same accumulation rate for the snow as in Exercise 11. Find the force exerted by the engine at $t = 2$ seconds.

17. Show that the velocity v of an object that has fallen a distance x from rest satisfies the equation $v^2 = 2xg$. (**Hint:** Velocity increases at a constant rate from 0 to v, so the distance x can be calculated as the product of the average velocity and time: $x = v_{ave} \cdot t = \left[(0+gt)/2\right]t = \frac{1}{2}gt^2$.)

18. Derive the result of Exercise 17 in an alternative way, using the fact that if air resistance is ignored, the potential energy of an object with mass m at altitude x, which is calculated as $E_p = mgx$, is turned into kinetic energy upon impact, which is calculated as $E_{kin} = \frac{1}{2}mv^2$.

19. In an attempt to escape from a predator, a small fish is swimming vertically downward at a rate of $75\ \text{cm/s}$. Find the rate of change of water pressure around the fish. Express your answer in atm/s. (**Hint:** Use the fact that underwater pressure at a depth of d meters is approximately $P(d) = 1 + 0.097d$ standard atmospheres (atm), where 1 atm = 101.325 kPa.)

20. Hydrogen may be obtained from water by a process called electrolysis, according to the process $2H_2O \rightarrow 2H_2 + O_2$. If we measure the production of hydrogen at $2.5\ \text{mol}/(\text{L}\cdot\text{h})$ (i.e., 2.5 moles of hydrogen per liter per hour), what will be the concentration of the newly obtained oxygen in 3 hours?

21. The combustion of ammonia gas (NH_3) produces nitrogen and water according to the process $4NH_3 + 3O_2 \rightarrow 2N_2 + 6H_2O$. Supposing that the rate of combustion is $0.5\ \text{mol}/(\text{L}\cdot\text{s})$, what is the rate of the production of water? How many milliliters of water are produced in two seconds? (**Hint:** Use the fact that the approximate molar mass of hydrogen is 1 g, while that of oxygen is 16 g. Also, the mass of 1 mL of water is 1 g.)

22. Magnesium is a flammable metal, and because of its bright light it has traditionally been used in camera flashes, illumination of mine shafts, fireworks, and flares. The reaction itself is described by $2Mg + O_2 \rightarrow 2MgO$. If magnesium burns in a chamber at an initial rate of $1.5\ \text{g/s}$, find the rate (in $\text{mol}/(\text{L}\cdot\text{s})$) at which the concentration of O_2 is decreasing in the chamber. (**Hint:** The approximate molar mass of magnesium is 24 g.)

23–29 Use the technique of linearization to determine the answer.

23. A manufacturer of small remote-controlled cars found its weekly revenue to be $R(x) = 160x - 0.3x^2$ dollars when x units are produced and sold.

a. Use marginal revenue to estimate the extra revenue when production is increased from 12 to 13 units.

b. Use the revenue function to calculate the actual revenue increase. Compare your answers.

24. Suppose the monthly cost of producing x units of a particular commodity is $C(x) = 75x^2 + 200x + 5100$ dollars, while the revenue function is $R(x) = 32x(120 - x)$.

a. Use marginal cost to find the added expense in increasing production from 5 to 6 units.

b. Use marginal revenue to estimate the revenue generated by raising production from 5 to 6 units.

c. Find the actual increases in cost and revenue by producing and selling the sixth unit, and compare these numbers to your estimations from parts a. and b.

25. Repeat Exercise 24 for $C(x) = \frac{486}{7}x^2 + 98x + 120$ and $R(x) = -19.5x^2 + 2526x + 442$.

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26. A manufacturer models the total cost of producing n hundreds of a particular pocket calculator by the function $C(n) = \frac{1}{900}n^2 + 2500n + 3100$ dollars. Market research shows that all products will be sold at the price of $p(n) = \frac{2}{5}\left(120 - \frac{1}{4}n\right)$ dollars per calculator.

a. Use the marginal cost function to estimate the cost of raising the level of production from 1000 to 1100 calculators.

b. Use the marginal profit function to estimate the additional profit if the level of production is raised from 1000 to 1100 calculators.

c. Find the actual increases in cost and profit when production is raised as in parts a. and b., and compare these values with your estimates obtained in the previous parts.

27. A lumber company estimates the cost of producing x units of a product to be $C(x) = 0.1x^2 + 2250x + 1450$ dollars, while the price of each unit has to be $p(x) = 50(128 - 0.2x)$ in order to sell all x units. However, seasonal supply of raw materials makes x dependent on time so that $x(t) = -0.35(t-6)^2 + 192$, where t is measured in months. Use the value of the marginal profit at $t = 2$ to estimate the change in profit during the third month.

28. Repeat Exercise 27 if $x(t) = 185 + 10\sin^2\left(\frac{\pi}{6}t\right)$.

29. A child playing on the beach is pouring sand from a bucket, forming a sand cone that is growing in such a way that its height is always half of the radius of its circular base. Estimate the change in volume of the sand cone as its height grows from 3 to 4 inches.

30–39 Slightly generalize the technique of linearization to find the answer. For cases where Δx is not 1, estimate the change in a function $f(x)$ by $\Delta f = f'(x)\Delta x$.

30. Suppose the cost of manufacturing x units of a certain commodity was found to be $C(x) = 35x^2 + 20x + 780$ dollars, and suppose the current production level is 50 units. Use linearization to estimate how the cost changes if production is raised to 50.25 units.

31. Repeat Exercise 30 with an increase in production from 13 to 13.5 units.

32. Repeat Exercise 24 with an increase in production from 6 to 6.75 units.

33. Use marginal analysis to estimate the changes in cost and profit of Exercise 26 when production is decreased from 1000 to 950 calculators. Then find the actual changes and compare them with your estimates.

34. An ice cube with a side length of 1.5 inches starts melting in such a way that its sides are decreasing by 0.1 inches per minute. Use linearization to estimate the change in the cube's volume during the second minute.

35. According to MRI scans, a benign tumor in a patient had a radius of 1.6 cm when it was first discovered, and it is growing by 1 mm each month. Assuming the tumor is spherical, estimate the change in its volume during the first week.

36. The daily output of a small factory is $n(I) = 50\sqrt{I}$ units, where I is the owner's investment measured in dollars. If the current investment is \$100,000, use linearization to estimate how much additional capital is needed to increase the daily output by 5%.

37. Suppose that $w(t)$, the number of workers at a certain factory at time t, has been decreasing at a rate of 2.5 percent, relative to the size of the workforce, due to a recent recession. At the same time, however, the workers' average productivity $p(t)$ has been increasing by 4 percent due to extra training and the inherent fear of a job loss. Find the change in gross productivity. (**Hint:** See Example 5.)

38. A factory outputs $n(I) = 200\sqrt{I(t)}\sqrt[3]{g(t)}$ units daily, where I is the total investment measured in dollars and g stands for gross productivity. If the total investment is decreasing by 1 percent while the gross productivity is increasing by 2.1 percent (both in the relative sense), find the relative change in the daily output of the factory.

39. Repeat Exercise 38 if $I(t)$ is decreasing by 2 percent, the number of workers is decreasing by 3 percent, but worker productivity is increasing by 5.1 percent. (**Hint:** Recall the equation from Example 5: $g(t) = w(t)p(t)$.)

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