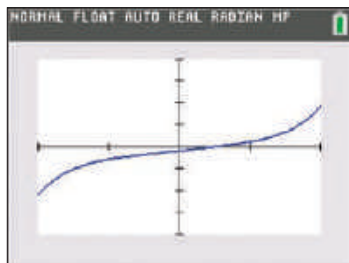


Figure 14

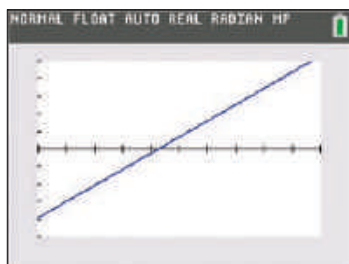
It is clear from Figure 14 that the root is located between 0.442 and 0.443, so we have achieved the desired accuracy; the first two digits after the decimal point are correct. We conclude that the root is  $c \approx 0.44$ . (Note that the above accuracy could actually have been achieved from carefully eyeballing the graph in Figure 13, while the graph in Figure 14 says even more: we might guess that the root is in fact very close to 0.4426.)

Note that it is also possible to estimate the root by appropriately zooming in on the graph using a graphing calculator (see Figure 15).

Finally, we use the **NSolve** command of *Mathematica* to approximate the solution of  $f(x) = 0$ . The screenshot in Figure 16 shows the feedback we receive from the software. The appearance of complex roots shows the power of the software, but we can ignore them for now and focus on the sole real root. *Mathematica* approximates it as  $c \approx 0.442558$ .



$f(x) = x^5 + 9x - 4$   
on  $[-2, 2]$  by  $[-100, 100]$



$f(x) = x^5 + 9x - 4$   
on  $[0.4, 0.5]$  by  $[-0.5, 0.5]$

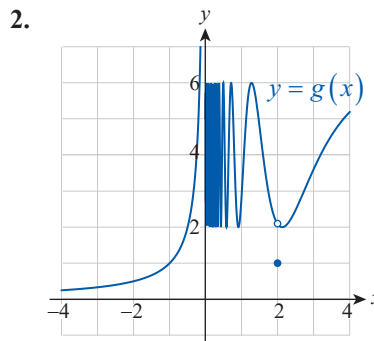
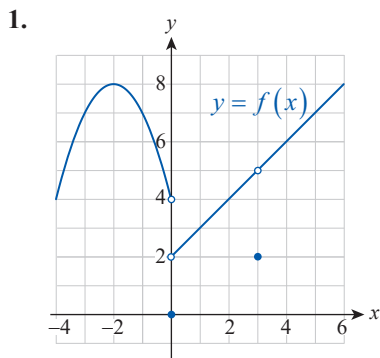
Figure 15 Zooming In on a Root with a Graphing Calculator

```
In[1]:= NSolve[x^5 + 9 x - 4 == 0, x]
Out[1]= {{x -> -1.32389 - 1.2337 i},
         {x -> -1.32389 + 1.2337 i}, {x -> 0.442558},
         {x -> 1.10261 - 1.24271 i}, {x -> 1.10261 + 1.24271 i}}
```

Figure 16

## 2.5 Exercises

1–2 Find all points of continuity as well as all points of discontinuity for the given function. For any discontinuities, identify those of the three continuity criteria that fail to hold.



3. Sketch a graph of a function (a formula is not necessary) that has a removable discontinuity at  $x = -1$ , a jump discontinuity at  $x = 2$ , but is right-continuous at 2. (Answers will vary.)
4. Sketch a graph of a function that has an infinite discontinuity at  $x = 0$  and an oscillating discontinuity at  $x = 5$  so that it is still left-continuous at 5. (Answers will vary.)

**5–29** Find and classify the discontinuities (if any) of the function as removable or nonremovable.

5.  $f(x) = \frac{1}{x}$                       6.  $g(x) = \frac{-2}{x-3}$
7.  $h(x) = \frac{x^2-9}{x-3}$                 8.  $k(x) = \frac{x^2-2x}{x^2+5x-14}$
9.  $u(x) = \frac{x^2-9}{x-2}$                 10.  $v(x) = \frac{x-1}{x^2+2x-3}$
11.  $w(x) = \begin{cases} x+1 & \text{if } x \leq 0 \\ \frac{1}{2}x^2+1 & \text{if } x > 0 \end{cases}$
12.  $f(x) = \begin{cases} \frac{1}{2}x-2 & \text{if } x \leq 4 \\ x^3+1 & \text{if } x > 4 \end{cases}$
13.  $g(x) = \begin{cases} \tan x & \text{if } x < \frac{\pi}{2} \\ \cos x & \text{if } x \geq \frac{\pi}{2} \end{cases}$
14.  $h(x) = \begin{cases} \cos x & \text{if } x \leq 0 \\ \tan x+1 & \text{if } x > 0 \end{cases}$
15.  $F(u) = \frac{u-4}{\sqrt{u}-2}, u \geq 0$
16.  $G(s) = \frac{s}{\sqrt{s+4}-2}, s \geq -4$
17.  $H(t) = \frac{t}{\sqrt{t^2+2}}$                 18.  $u(x) = \cos \frac{1-x^2}{1-x}$
19.  $v(t) = |\sin t|$                 20.  $K(x) = |x+2| + |x-1|$
21.  $F(t) = \frac{t}{t^2-1}$                 22.  $G(t) = \frac{t}{t^2+1}$
23.  $H(x) = |x+2|$                 24.  $k(x) = \frac{|x-4|}{x-4}$
25.  $s(x) = \llbracket x+2 \rrbracket$                 26.  $t(x) = 4 - \llbracket x \rrbracket$
27.  $u(z) = \llbracket z^2 \rrbracket$                 28.  $v(x) = x \llbracket x \rrbracket$
29.  $w(x) = x \llbracket \frac{1}{x} \rrbracket$

**30–33** Use the  $\varepsilon$ - $\delta$  definition to prove that the function is continuous.

30.  $f(x) = \frac{1}{x}$                               31.  $g(x) = 3x-2$
32.  $F(x) = x^3$                             33.  $G(x) = \sqrt{x}$

**34–39** Use the theorems of this section to discuss the continuity of the function.

34.  $F(x) = \sqrt{\frac{x}{x^2+7x+12}}$
35.  $G(x) = \sqrt{\frac{x^4-x^3-11x^2+9x+18}{2x^3+x}}$
36.  $H(x) = \cos \frac{2 \ln(x-3)+1}{\sqrt[3]{x^2-2x-15}}$
37.  $f(x) = \arctan \frac{x}{\sqrt{3-x^2}}$
38.  $g(x) = \ln(\arcsin(\pi x+1))$
39.  $h(x) = \frac{\csc(\pi x+1)}{\sin(\pi e^{x+2})}$

40. Prove the alternate formulation of continuity, that is, the statement that a function  $f$  is continuous at the point  $c$  if and only if  $\lim_{h \rightarrow 0} f(c+h) = f(c)$ .

**41–44** Use the alternate formulation of continuity to prove that the function is continuous.

41.  $f(x) = 3x-5x^2$                       42.  $g(x) = \cos x$
43.  $h(x) = \tan x$                         44.  $k(x) = e^x$

**45–50** Identify the removable discontinuities and define the continuous extension of the function.

45.  $f(x) = \frac{x^2+x-12}{x-3}$
46.  $g(x) = \frac{x^3-2x^2-x+2}{x^2-3x+2}$
47.  $h(x) = \frac{x-1}{\sqrt{x}-1}$                       48.  $F(x) = \frac{\sqrt{x+1}-2}{x-3}$
49.  $G(x) = 2^{-1/x^2}$                       50.  $H(x) = x \cos \frac{\pi}{x}$

**51–54** Discuss the continuity of the function on the given closed interval.

51.  $S(x) = \sqrt{16 - x^2}$  on  $[-4, 4]$

52.  $T(x) = \left\lfloor \frac{x}{3} \right\rfloor$  on  $[0, 3]$

53.  $U(x) = \begin{cases} \frac{1}{x^2 - 9} & \text{if } |x| < 3 \\ 0 & \text{if } |x| = 3 \end{cases}$  on  $[-3, 3]$

54.  $V(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$  on  $\left[0, \frac{1}{\pi}\right]$

**55–58** Find the value of  $a$  (or the values of  $a$  and  $b$ , where applicable) such that  $f$  is continuous on the entire real line.

55.  $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ ax & \text{if } 0 < x < 1 \\ 2x + 3 & \text{if } x \geq 1 \end{cases}$

56.  $f(x) = \begin{cases} x^3 & \text{if } x \leq 3 \\ ax^2 & \text{if } x > 3 \end{cases}$

57.  $f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ ax + b & \text{if } 1 \leq x \leq 3 \\ (x - 3)^2 + 2 & \text{if } x > 3 \end{cases}$

58.  $f(x) = \begin{cases} \cos x & \text{if } x \leq 0 \\ -(x - a)^2 + b & \text{if } 0 < x < 2 \\ \frac{1}{2}x - 2 & \text{if } x \geq 2 \end{cases}$

59. Prove that if  $f(x)$  is continuous and  $f(c) > 0$ , then there is a  $\delta > 0$  such that  $f(x) > 0$  for all  $x \neq c$  in the interval  $(c - \delta, c + \delta)$ .

60. Prove that the Dirichlet function

$$\xi(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every real number.

61. Prove that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at the single point  $c = 0$ .

62. Prove that if the functions  $f$  and  $g$  are both continuous on  $\mathbb{R}$  and they agree on the rationals (i.e.,  $f(x) = g(x)$  for all  $x \in \mathbb{Q}$ ), then  $f = g$ .

**63–68** Decide whether the Intermediate Value Theorem applies to the given function on the indicated interval. If so, find  $c$  as guaranteed by the theorem. If not, find the reason.

63.  $f(x) = -x^2 + x + 3$  on  $[0, 3]$ ;  $f(c) = 1$

64.  $g(x) = 2x^3 - x^2 - 1$  on  $[-1, 2]$ ;  $g(c) = 0$

65.  $h(x) = \frac{x}{x+2}$  on  $[0, 4]$ ;  $h(c) = \frac{1}{2}$

66.  $F(x) = \frac{2x}{x-1}$  on  $[0, 2]$ ;  $F(c) = 2$

67.  $G(x) = \lfloor x - 2 \rfloor$  on  $[-2, 2]$ ;  $G(c) = -\frac{1}{2}$

68.  $H(x) = \sin \frac{3x+2}{2}$  on  $\left[-\frac{2}{3}, \frac{\pi-2}{3}\right]$ ;  $H(c) = \frac{1}{2}$

**69–74** Use the Intermediate Value Theorem to prove that the given equation has a solution on the indicated interval.

69.  $x^3 - 7.5x^2 + 1.2x + 1 = 0$  on  $[-1, 0]$

70.  $2x^3 + x + 10 = 0$  on  $[-2, 1]$

71.  $\cos x = x^2$  on  $[0, \pi]$

72.  $\ln x - \sqrt{x-2} = 0$  on  $[2, 5]$

73.  $\frac{5}{x^2 + 2} = 1$  on  $[-3, -1]$

74.  $\cot \frac{\pi x}{4} - \frac{x}{x+2} = -\frac{1}{2}$  on  $[1, 3]$

75. Suppose that the outside temperature in Columbia, SC, on a summer morning at 7:00 a.m. is 74 °F, and it shoots up to 98 °F by 1:00 p.m. Assuming that temperature changes continuously, prove that sometime between 7:00 a.m. and 1:00 p.m. the temperature was exactly 88.35 °F.

76. Prove that if  $f$  is continuous and never 0 on the interval  $[a, b]$ , then either  $f(x) > 0$  for every  $x$  in  $[a, b]$ , or  $f(x) < 0$  for every  $x$  in  $[a, b]$ .

77.\* (Existence of  $n^{\text{th}}$  roots) Prove that if  $b$  is a positive real number and  $n$  is a positive integer, then there is a positive real number  $c$  such that  $c^n = b$ . (**Hint:** Consider the continuous function  $f(x) = x^n$  on the interval  $[0, b+1]$ .)

78.\* Prove that a circle of diameter  $d$  has a chord of length  $c$  for every number  $c$  between 0 and  $d$ .

79. Use the function

$$f(x) = \begin{cases} \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

to prove that the converse of the Intermediate Value Theorem is false; in other words, a function may possess the Intermediate Value Property without being continuous.

80.\* (Fixed Point Theorem) Prove that if the function  $f : [a, b] \rightarrow [a, b]$  is continuous, then there is a number  $c$  in  $[a, b]$  with  $f(c) = c$  (i.e.,  $c$  is “fixed,” or “not being moved,” by  $f$ ).

81.\* A hermit leaves his hut at the foot of a mountain one day at 6:00 a.m. and sets out to climb all the way to the top. He arrives at 6:00 p.m. and realizes that it is too late to go back, so he sets up camp for the night. At 6:00 a.m. the following day, he starts hiking back to his hut, taking the exact same route as the day before. This time, however, it is mostly downhill, so he makes much better time and arrives home at 2:00 p.m. Prove that there is a point along the hermit’s route that he passed at exactly the same time on both days. (**Hint:** Apply the Intermediate Value Theorem or the Fixed Point Theorem.)

82. For certain international calls, a phone company charges 31 cents for the first minute and 10 cents for each additional minute or any fraction thereof. Graph the cost as a function of time, find a formula for it, and discuss the significance of its discontinuities. (**Hint:** Use the greatest integer function to construct your answer.)

83.\* If  $\Delta t$  denotes the length of the time interval between two events as measured by an observer on a spaceship moving at speed  $v$ , and  $\Delta T$  is the length of the same time interval as measured from Earth, then the formula relating the two quantities is given by

$$\Delta T = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where  $c$  is the speed of light. This phenomenon is called *time dilation*, and it follows from the theory of relativity. In essence, it says that a clock moving at speed  $v$  relative to an observer is perceived by the same observer to run slower.

- Explain why we don’t normally notice the time dilation effect in everyday life.
- What is the significance of the discontinuity of  $\Delta T$  (as a function of  $v$ )?

## Concept Check

84–88 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

- If  $f$  is both left- and right-continuous at  $c$ , then  $f$  is continuous at  $c$ .
- Any function  $f$  has an interval  $(a, b)$  on which it is continuous.
- If  $c$  is a discontinuity of  $f$ , but  $f$  does not have a vertical asymptote at  $c$ , then  $c$  is a removable or jump discontinuity.
- If  $\lim_{x \rightarrow c} f(x) = L$ , and  $f(c) = L$ , then  $f$  is continuous at  $c$ .
- If  $c$  is a discontinuity of  $f$ , but  $\lim_{x \rightarrow c^+} f(x)$  exists, then  $c$  is a removable or jump discontinuity.

## 2.5 Technology Exercises

- Use a graphing utility to solve the equations given in Exercises 69–74 to four decimal places.
- Use a graphing utility to graph the functions of Exercises 34–39 and explain how the graphs support your discussions of continuity in the aforementioned exercises.