We will end with one last limit theorem that will prove useful in some derivations to follow. (See Appendix E for a proof.)

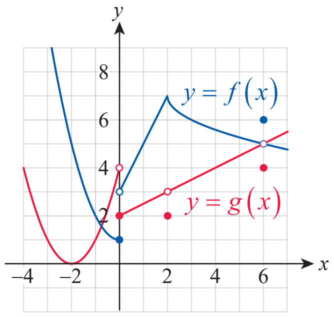
Theorem Upper Bound Theorem

If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at c itself, and if the limits of f and g both exist at c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

2.4 Exercises

1–2 Use the graph to find the given limit.



1. **a.** $\lim_{x\to 0^+}\left[g(x)-2f(x)\right]$ **b.** $\lim_{x\to 2^+}\left[g(x)f(x)\right]$

2. **a.** $\lim_{x\to 6}\left[g(x)+f(x)\right]$ **b.** $\lim_{x\to 0^-}\frac{f(x)}{2g(x)}$

3–20 Use appropriate limit laws to evaluate the given limit.

3. $\lim_{x\to 4} 5$

4. $\lim_{x\to 4} 5x$

5. $\lim_{x\to 3}(2x+1)$

6. $\lim_{x\to 1/2}(3-4x)$

7. $\lim_{x\to -3} x^2$

8. $\lim_{x\to -2}\left(-x^5\right)$

9. $\lim_{x\to 3}\left(2x^2-x+7\right)$

10. $\lim_{x\to -1}\left(3+x-\frac{5}{2}x^2\right)$

11. $\lim_{x\to 1/2}\left(2x^3-3x^2+x-4\right)$

12. $\lim_{x\to -2}\left(3x^3-x^5\right)$

13. $\lim_{x\to 1}\frac{3x-7}{x+1}$

14. $\lim_{x\to -1}\frac{5x+3}{x^2-x}$

15. $\lim_{x\to 3}\left(\frac{4x}{11x-x^3}\right)^{1/3}$

16. $\lim_{t\to 1}\left(\frac{2t+t^3}{3t^2+1}\right)^{3/2}$

17. $\lim_{x\to -2}\sqrt[3]{5x^4-x^3+3x^2+2x+4}$

18. $\lim_{x\to 4}\sqrt{x^4+2x^2+1}$

19. $\lim_{x\to -3}\left(\frac{x^4-5x}{x^3+2x^2-4x}\right)^{4/5}$

20. $\lim_{x\to -5}\sqrt[3]{\left(x^4+2x^3+x^2\right)^2}$

21–44 Use algebra to evaluate the given limit.

21. $\lim_{x\to 6}\frac{x^2-36}{x-6}$

22. $\lim_{x\to -7}\frac{x+7}{x^2-49}$

23. $\lim_{x\to 3}\frac{3-13x+4x^2}{x-3}$

24. $\lim_{x\to 4}\frac{x^4-256}{x^2-16}$

25. $\lim_{x\to 5}\frac{2x^3-7x^2-14x-5}{x^2-25}$

26. $\lim_{x\to 2}\frac{x^3-8}{x^3-2x^2+2x-4}$

27. $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$

28. $\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}$

29. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

30. $\lim_{x\to 0}\frac{\frac{1}{4+x}-\frac{1}{4}}{x}$

31. $\lim_{x\to 2}\frac{\frac{1}{3}-\frac{1}{1+x}}{x-2}$

32. If $f(x)=x^2$, find $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$.

33. If $g(x)=x^2-2$, find $\lim_{h\to 0}\frac{g(3+h)-g(3)}{h}$.

34. If $k(x)=1-x+x^2$, find $\lim_{h\to 0}\frac{k(2-h)-k(2)}{h}$.

35. If $p(x)=x^3+x$, find $\lim_{x\to 1}\frac{p(x)-p(1)}{x-1}$.

36. If $F(x)=\frac{1}{x}$, find $\lim_{x\to 1/2}\frac{F(x)-F\left(\frac{1}{2}\right)}{x-\frac{1}{2}}$.

37. $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

38. $\lim_{x\to 3}\frac{2x^2-3x-9}{x^2-9}$

39. $\lim_{x\to 2}\frac{x^4-16}{3x^2-5x-2}$

40. $\lim_{x\to -3}\frac{\frac{1}{3}+\frac{1}{x}}{x^3+27}$

41. $\lim_{x\to 2}\frac{3-\sqrt{x+7}}{x-2}$

42. $\lim_{x\to 8}\frac{8-x}{\sqrt[3]{x}-2}$

43. $\lim_{y \to 0}\left(\frac{1}{y}+\frac{1}{y^2-y}\right)$ **44.** $\lim_{x \to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$

45–50 Use $\lim_{x \to c} f(x)=3$ and $\lim_{x \to c} g(x)=-2$ to find the limit.

45. $\lim_{x \to c}\left[2f(x)-g(x)\right]$

46. $\lim_{x \to c}\frac{4f(x)+3g(x)}{f(x)-\frac{1}{2}g(x)}$

47. $\lim_{x \to c}\sqrt{\left[f(x)\right]^4+10\left[g(x)\right]^2}$

48. $\lim_{x \to c}\left(\left[f(x)-1\right]^2\sqrt[3]{g(x)}\right)$

49. $\lim_{x \to c}\left(\left[f(x)\right]^2+(x-2)g(x)\right)$

50. $\lim_{x \to c}\left(\frac{f(x)+g(x)}{\left[g(x)\right]^2}\right)^{3/2}$

51–58 Use the limit laws to find the one-sided limit.

51. $\lim_{x \to 0^+}\frac{x}{|x|}$

52. $\lim_{x \to 0^-} \operatorname{sgn} x \cos x$, where $\operatorname{sgn} x=\begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$

53. $\lim_{x \to 2^+}\frac{\sqrt{x-2}}{3x+1}$ **54.** $\lim_{x \to 1^-}\sqrt{1-x^2}$

55. $\lim_{x \to (1/3)^-}\frac{\sqrt{1-3x}}{6x+5}$

56. $\lim_{x \to 1^+}\left([\![x]\!]-x\right)$ (See the definition of $f(x)=[\![x]\!]$ in Section 1.2, before Exercises 43–45.)

57. $\lim_{x \to 2^+}[\![x]\!]e^x$ **58.** $\lim_{x \to -1^-}\frac{[\![x]\!]\left(2x^2+1\right)}{x+3}$

59–64 Use the Squeeze Theorem to prove the limit claim.

59. $\lim_{x \to 0} x^2 \sin\frac{1}{x}=0$ **60.** $\lim_{x \to 0}|x|\cos x=0$

61. $\lim_{x \to \infty}\frac{\cos x}{x}=0$ **62.** $\lim_{x \to -\infty} e^x \sin x=0$

63. $\lim_{x \to 0^+} x^{3/2}e^{\cos(1/x)}=0$ **64.** $\lim_{x \to \infty}\frac{\sin^2 x+1}{2+x}=0$

65. Provide a rigorous proof of the limit claim $\lim_{x \to c} 1=1$. (**Hint:** Use the fact that for the constant 1 function, $f(x)=1$ for all x, so in particular, if an $\varepsilon>0$ is given, $|f(x)-1|=|1-1|=0$, which makes the choice of δ "easy.")

66. Provide a rigorous proof of the limit claim $\lim_{x \to c} x=c$. (**Hint:** Since $f(x)=x$ in this problem, for a given $\varepsilon>0$ we need to ensure that $|f(x)-c|=|x-c|<\varepsilon$ as long as $0<|x-c|<\delta$. This observation makes the choice of δ obvious.)

67. Use Exercise 66 and the basic limit laws to prove the Polynomial Substitution Law. (**Hint:** From Exercise 66 and a repeated application of the Product Law, it follows that $\lim_{x \to c} x^k=c^k$. As a next step, from the Constant Multiple Law we can conclude that if $a \in \mathbb{R}$, $\lim_{x \to c} ax^k=ac^k$. From the above claim, a repeated application of the Sum Law will yield the result for a general polynomial.)

68. Use Exercise 67 and the basic limit laws to prove the Rational Function Substitution Law.

69. Combine the Positive Integer Power Law and the Positive Integer Root Law to prove the Rational Power Law. (**Hint:** Assuming first that both m and n are positive, we can write $\lim_{x \to c}\left[f(x)\right]^{m/n}=\lim_{x \to c}\left[\left[f(x)\right]^{1/n}\right]^m=\lim_{x \to c}\left[\sqrt[n]{f(x)}\right]^m$. Now use the Positive Integer Power Law followed by the Positive Integer Root Law to obtain that the above limit is equal to $\left[\lim_{x \to c}\sqrt[n]{f(x)}\right]^m=\left[\sqrt[n]{\lim_{x \to c} f(x)}\right]^m$, from which the result follows. If m is negative, note that $\left[f(x)\right]^{m/n}=1\big/\left[f(x)\right]^{-m/n}$, where $-m$ is positive. Thus if we use the Quotient Law along with the previous argument, we obtain

$$\begin{aligned}\lim_{x \to c}\left[f(x)\right]^{m/n} &= \lim_{x \to c}\frac{1}{\left[f(x)\right]^{-m/n}} \\ &= \frac{1}{\lim_{x \to c}\left[f(x)\right]^{-m/n}} \\ &= \frac{1}{\left[\lim_{x \to c} f(x)\right]^{-m/n}},\end{aligned}$$

from which the result follows.)

70. Let $D(x)=\begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$.

Does $\lim_{x \to 0} D(x)$ exist? Prove your answer.

71. Let $F(x)=\begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$.

Does $\lim_{x \to 0} F(x)$ exist? Prove your answer.

72.* Prove that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = K$, then $L = K$. In words, prove that if the limit of f exists at c, then the limit is unique.

73.* Prove that if n and m are positive integers, then $\lim_{x\to 1} \frac{x^n - 1}{x^m - 1} = \frac{n}{m}$.

74.* Prove that if $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} |f(x)| = 0$.

75.* Prove that if $\lim_{x\to c} f(x) = 0$ and $g(x)$ is such that $|g(x)| \le M$ for some number M (such functions are called bounded), then $\lim_{x\to c}[f(x)g(x)] = 0$.

76.* Prove that in Exercise 75, it is sufficient to require the boundedness of g only on an interval around c (except at c itself).

77.* By finding functions f and g such that $\lim_{x\to c} f(x) = 0$ but $\lim_{x\to c}[f(x)g(x)] \ne 0$, show that it is necessary to impose a boundedness condition on g in Exercise 75.

78.* Give examples of f and g to show that **a.** the existence of $\lim_{x\to c}[f(x)+g(x)]$ does not imply the existence of $\lim_{x\to c} f(x)$ and **b.** the existence of $\lim_{x\to c}[f(x)g(x)]$ does not imply the existence of $\lim_{x\to c} f(x)$.

79. A *concave spherical mirror* is a part of the inside of a sphere, silvered to form a reflective surface. The radius r of the sphere is called the mirror's *radius of curvature*. If the size of such a mirror is small relative to its radius of curvature, light rays parallel to its principal axis are reflected through approximately a single point, called *focus*. In the following illustration, C denotes the center, F_d is the focus, while d is the distance between the incoming ray and the principal axis. Note that according to the Law of Reflection, the incoming and reflected rays make the same size angle α with the radius $\overline{CR}$ (this radius is called *normal* to the mirror surface). One way to determine the *focal length* (the distance between the mirror and the focus along the principal axis) is to find the limiting position of F_d as $d \to 0$. Noting that the triangle $\triangle CRF_d$ is isosceles, by similarity we obtain $\frac{CF_d}{(r/2)} = \frac{r}{\sqrt{r^2 - d^2}}$.

Use this observation to express CF_d and then determine the focal length of the spherical mirror by taking the limit as $d \to 0$.