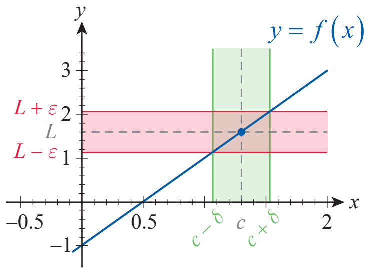
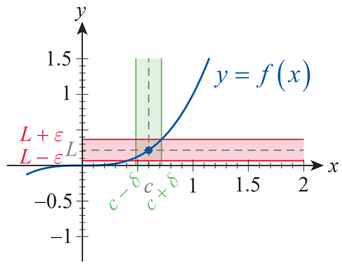
2.3 Exercises

1–4 Use the graph to estimate δ corresponding to the given ε satisfying the ε-δ definition of $\lim_{x\to c} f(x) = L$.

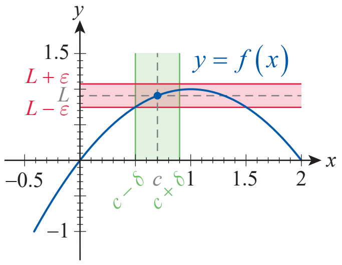
1.



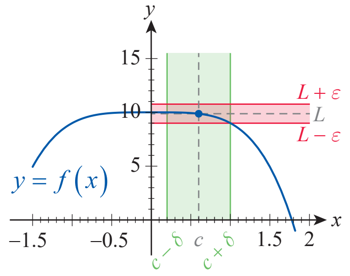
2.



3.



4.



5–10 Calculus students gave the following definitions for the existence of a limit of $f(x)$ at c. Find and correct any errors.

5. "$\lim_{x\to c} f(x)$ exists if for any $\varepsilon > 0$ and real number L there is a $\delta > 0$ such that $0 < |x - L| < \delta$ implies $|f(x) - c| < \varepsilon$."

6. "$\lim_{x\to c} f(x)$ exists and equals L if for any $\varepsilon > 0$ and $\delta > 0$ whenever $0 < |x - c| < \varepsilon$, we have $|f(x) - L| < \delta$."

7. "If there is a real number L such that for an $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \neq c$, we have $|f(x) - L| < \varepsilon$, we say that the limit of the function at c is L."

8. "We say that $\lim_{x\to c} f(x) = L$, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$."

9. "We say that $\lim_{x\to c} f(x) = L$, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $0 \le |x - c| \le \delta \Rightarrow |f(x) - L| \le \varepsilon$."

10. "If the real number L is such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$, we say that $\lim_{x\to c} f(x) = L$."

11–20 Find a $\delta > 0$ that satisfies the limit claim corresponding to $\varepsilon = 0.1$, that is, such that $0 < |x - c| < \delta$ would imply $|f(x) - L| < 0.1$.

11. $\lim_{x\to 2}(5x - 1) = 9$

12. $\lim_{x\to 1}(3x + 1) = 4$

13. $\lim_{x\to -1}(-x + 2) = 3$

14. $\lim_{x\to 6}\left(4 - \frac{x}{2}\right) = 1$

15. $\lim_{x\to 0} x^2 = 0$

16. $\lim_{x\to 8} \sqrt[3]{x} = 2$

17. $\lim_{x\to 1} \frac{1}{x} = 1$

18. $\lim_{x\to 0} e^x = 1$

19. $\lim_{x\to 1} \ln x = 0$

20. $\lim_{x\to 0} \cos x = 1$

21–26 Find a number N that satisfies the limit claim corresponding to $\varepsilon = 0.1$, that is, such that $x > N$ (or $x < N$, as appropriate) would imply $|f(x) - L| < 0.1$.

21. $\lim_{x\to\infty} \frac{1}{x} = 0$

22. $\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 2x} = 1$

23. $\lim_{x\to -\infty} \frac{x + 1}{x} = 1$

24. $\lim_{x\to\infty} \frac{2x}{\sqrt{x^2 + x}} = 2$

25. $\lim_{x\to -\infty} e^x = 0$

26. $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

27–32 For the given function $f(x)$, find a $\delta > 0$ corresponding to $M = 100$, that is, such that $0 < |x - c| < \delta$ would imply $f(x) > 100$ (let $N = -100$ if the limit is $-\infty$, in which case $0 < |x - c| < \delta$ should imply $f(x) < -100$).

27. $\lim_{x\to 0} \frac{1}{x^2} = \infty$

28. $\lim_{x\to 0^+} \frac{1}{x} = \infty$

29. $\lim_{x\to -1} \frac{-1}{(x+1)^2} = -\infty$

30. $\lim_{x\to 0^+} \ln x = -\infty$

31. $\lim_{x\to(\pi/2)^-} \tan x = \infty$

32. $\lim_{x\to 0^+} \csc x = \infty$

33–46 Use the ε-δ definition to prove the limit claim. (**Hint:** See Examples 3 and 4 for guidance as you work through these exercises.)

33. $\lim_{x\to 1}(2x+3) = 5$

34. $\lim_{x\to 7} x = 7$

35. $\lim_{x\to c} a = a$

36. $\lim_{x\to 4}\left(\frac{1}{4}x + 1\right) = 2$

37. $\lim_{x\to 0}\left(\frac{1}{2} - 4x\right) = \frac{1}{2}$

38. $\lim_{x\to 9}\left(5 - \frac{x}{3}\right) = 2$

39. $\lim_{x\to 1} x^3 = 1$

40. $\lim_{x\to 0} x^2 = 0$

41. $\lim_{x\to 0} \frac{1}{2}|x| = 0$

42. $\lim_{x\to -2} |x+2| = 0$

43. $\lim_{x\to 1^+} \sqrt{x-1} = 0$

44. $\lim_{x\to 0^-}\left(\sqrt[3]{x} + 1\right) = 1$

45. $\lim_{x\to 1}\left(x^2 + x\right) = 2$

46. $\lim_{x\to 3}\left(3x^2 - 9x + 5\right) = 5$

47–62 Give the formal definition of the limit claim. Then use the definition to prove the claim. (**Hint:** See Examples 5 and 6 for guidance as you work through these exercises.)

47. $\lim_{x\to\infty} \frac{1+x}{x} = 1$

48. $\lim_{x\to -\infty} \frac{2}{x^2} = 0$

49. $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

50. $\lim_{x\to\infty} \frac{1+3x^3}{x^3} = 3$

51. $\lim_{x\to -\infty} 2^x = 0$

52. $\lim_{x\to\infty}\left(e^{-x} - 1\right) = -1$

53. $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

54. $\lim_{x\to\infty}\left(2\arctan x\right) = \pi$

55. $\lim_{x\to 0^+} \frac{1}{x} = \infty$

56. $\lim_{x\to 0} \frac{1}{x^4} = \infty$

57. $\lim_{x\to -1} \frac{-1}{(x+1)^2} = -\infty$

58. $\lim_{x\to 0^+} \log x = -\infty$

59. $\lim_{x\to(\pi/2)^+} \tan x = -\infty$

60. $\lim_{x\to 0^+} \csc x = \infty$

61. $\lim_{x\to 2} \frac{-3}{(x-2)^2} = -\infty$

62. $\lim_{x\to -2^-} \frac{x+3}{x+2} = -\infty$

63–67 Decide whether the given limit exists. Prove your conclusion. (**Hint:** See Example 7 for guidance as you work through these exercises.)

63. $\lim_{x\to 0^+} \sin\frac{\pi}{x}$

64. $\lim_{x\to 0^+} x^2 \cos\frac{1}{x}$

65. $\lim_{x\to 0} \frac{|x|}{x}$

66.* $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

67.* $\lim_{x\to 0} g(x)$, where $g(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

68. Use ε and δ to state what $\lim_{x\to c} f(x) \neq L$ means.

69. A piston is manufactured to fit into the cylinder of a certain automobile engine. Suppose that the diameter of the cylinder is 82 mm and that the cross-sectional area of the piston is not allowed to be less than 99.89% of that of the cylinder. If both are perfectly round, what does this mean in terms of maximum tolerance for the clearance between the piston and the cylinder wall? (Be sure to identify which function and data take the roles of $f(x)$, c, ε, and δ in this problem.)

70. The tension in a stretched steel wire (in newtons, N) is calculated by the formula $F = E\frac{\Delta L}{L_0}A$, where $E = 2\times 10^{11}\ \text{N/m}^2$ is the elastic modulus (or Young's modulus) of steel, ΔL is the elongation, L_0 the original length, and A the cross-sectional area (in m^2). Suppose a 1-meter-long steel string of radius 1 millimeter is stretched by 2 millimeters when tuning a musical instrument.

a. Calculate the tension in the string caused by the above tightening.

b. If we are not allowed to overload the string by more than 100 N, what is the tolerance in the amount of stretching? (Be sure to identify the function and data taking the roles of c, ε, and δ in this problem.)

Concept Check

71–73 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

71. If $f(c) = L$, then as x approaches c, $\lim_{x\to c} f(x) = L$.

72. If $\lim_{x\to c} f(x)$ exists and equals L, then $f(c) = L$.

73. If $f(x) < g(x)$ for all $x \neq c$, and both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then $\lim_{x\to c} f(x) < \lim_{x\to c} g(x)$.

2.3 Technology Exercises

74–83 Use a graphing utility to estimate the given limit. By zooming in appropriately, find δ-values that correspond to $\varepsilon = 0.1$. (Answers will vary.)

74. $\lim_{x\to 5}\frac{x^2-5x+6}{x-2}$

75. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

76. $\lim_{x\to 3.5}\frac{x^2-6.25}{x+2.5}$

77. $\lim_{x\to 0}\frac{x-1}{\sqrt{x}-1}$

78. $\lim_{x\to 0}\frac{\sin 3x}{2x}$

79. $\lim_{x\to -\infty}\frac{2x+3}{\sqrt{x^2+1}}$

80. $\lim_{x\to \infty}\frac{\sqrt{9x^2+1}}{x-2}$

81. $\lim_{x\to -\infty}\frac{2x^2+1.5x-7}{\sqrt{x^4+1}}$

82. $\lim_{x\to \infty}\left(\sqrt{x^2+3x+5}-\sqrt{x^2+2x+1}\right)$

83. $\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x$

84–89 Use a graphing utility to locate a vertical asymptote of the given function. Then for such an asymptote $x = c$ find an appropriate value $\delta > 0$ such that $|x-c| < \delta \Rightarrow |f(x)| > 10$. (Answers will vary.)

84. $f(x) = \frac{x^2-7}{x^3+x+1}$

85. $f(x) = \frac{3x+1}{2x^4+x-5}$

86. $f(x) = \ln\frac{x^2}{x^2+1}$

87. $f(x) = \tan\left(\frac{1}{2}x+3\right)$

88. $f(x) = \csc(2x+1)$

89. $f(x) = \cot\left(\frac{1}{2}\cos x\right)$