

## 15.8 Exercises

**1–4** Verify the Divergence Theorem by showing the equality of the integrals  $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$  and  $\iiint_D \nabla \cdot \mathbf{F} dV$  for the given vector field  $\mathbf{F}$  on the solid  $D$ .

- $\mathbf{F}(x, y, z) = \langle -y, x, z \rangle$ , where  $D$  is the unit ball centered at the origin
- $\mathbf{F}(x, y, z) = \langle x, 2y, 3z \rangle$ , where  $D$  is the tetrahedron bounded by the coordinate planes and  $z = 1 - x - y$
- $\mathbf{F}(x, y, z) = \langle xz, xy, yz \rangle$ , where  $D$  is the cube given in Example 1b
- $\mathbf{F}(x, y, z) = \langle xz, 3yz, z^2 \rangle$ , where  $D$  is the solid bounded by the paraboloid  $z = 4 - x^2 - y^2$  and the  $xy$ -plane

**5–18** Use the Divergence Theorem to find the flux of the vector field  $\mathbf{F}$  over the surface of the given solid  $D$ . Consider cylindrical or spherical coordinates where appropriate.

- $\mathbf{F}(x, y, z) = \langle 2x - y^2, zx - y, z + e^{xy} \rangle$ , where  $D$  is the solid ball of radius 2, centered at the origin
- $\mathbf{F}(x, y, z) = \langle 3x - yz, 2y + e^{xz}, \cos 2y - 2z \rangle$ , where  $D$  is the solid bounded by the parabolic cylinder  $z = 1 - y^2$ , the  $xy$ -plane, and the planes  $x = 0$  and  $x = 1$
- $\mathbf{F}(x, y, z) = \langle x, 3y, z - y^2 \rangle$ , where  $D$  is the solid spherical shell  $4 \leq x^2 + y^2 + z^2 \leq 9$
- $\mathbf{F}(x, y, z) = \left| \langle x, y, z \rangle \right|^2 \langle x, y, z \rangle$ , where  $D$  is the cube given in Example 1b
- $\mathbf{F}(x, y, z) = \langle 3x - yz, 2y + e^{xz}, \cos 2y - 2z \rangle$ , where  $D$  is the solid cylinder bounded by  $x^2 + y^2 = 1$ , the  $xy$ -plane, and the plane  $z = 2$
- $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ , where  $D$  is the solid cylinder given in Exercise 9
- $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ , where  $D$  is the solid bounded by the paraboloid  $z = 1 - x^2 - y^2$  and the  $xy$ -plane
- $\mathbf{F}(x, y, z) = \langle y - 2x, e^{2xz}, z^2 - \tan^{-1}(xy) \rangle$ , where  $D$  is the tetrahedron with vertices at the origin,  $(2, 0, 0)$ ,  $(0, 4, 0)$ , and  $(0, 0, 1)$
- $\mathbf{F}(x, y, z) = \langle 6xy^2, 6x^2y, x \cos(y^2) \rangle$ , where  $D$  is the solid bounded by the paraboloid  $z = 4 - x^2 - y^2$  and the plane  $z = 3$

- $\mathbf{F}(x, y, z) = \langle x - 3z, y, 4z^2 \rangle$ , where  $D$  is the solid inside the cylinder  $x^2 + y^2 = 1$ , between the planes  $z = 1$  and  $z = 4 - x$
- $\mathbf{F}(x, y, z) = \langle xy - \sqrt{y^2 + z^2}, 2y + e^{z^3}, 3z - x^2 \cot y \rangle$ , where  $D$  is the solid inside the cylinder  $x^2 + y^2 = 1$ , between the planes  $z = 1$  and  $z = x + 4$
- $\mathbf{F}(x, y, z) = \langle 2x^2, ze^x, y - xz \rangle$ , where  $D$  is the solid cylindrical shell  $1 \leq x^2 + y^2 \leq 2$  between the  $xy$ -plane and  $z = 4$
- $\mathbf{F}(x, y, z) = \langle 3y^2z, y^3/3, x^2z \rangle$ , where  $D$  is the portion of the solid cone  $z = \sqrt{x^2 + y^2}$  between the planes  $z = 1$  and  $z = 3$
- \*  $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ , where  $D$  is the solid hemisphere  $z = \sqrt{1 - x^2 - y^2}$

**19.** Verify the Divergence Theorem for the vector field  $\mathbf{F}(x, y, z) = \langle z^2, 3y^2, 2yz \rangle$  and the solid bounded by the cylinder  $x^2 + z^2 = 1$ , the  $xz$ -plane, and  $y = 2$ .

**20.** If  $\mathbf{F}(x, y, z) = \langle z^2 \tan y, 3xe^z, y \sin 2xz \rangle$ ,  $S$ , and  $D$  satisfy the conditions of the Divergence Theorem, prove

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = 0.$$

- Use the Divergence Theorem to find the flux of the vector field  $\mathbf{F}(x, y, z) = \langle x^3y, y^2 - \sin z^2, xe^z \rangle$  over the surface of the box  $[0, 4] \times [0, 2] \times [0, 3]$ .
- Suppose the space region  $D$  and its boundary  $S$  with unit normal field  $\mathbf{n}$  meet the conditions of the Divergence Theorem, and let  $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ . Use the Divergence Theorem to show that the volume of  $D$ ,  $V(D)$  is equal to the following surface integral.

$$V(D) = \frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$$

- Use Exercise 22 to verify the formula for the volume of the box with side lengths  $a$ ,  $b$ , and  $c$ , respectively. (**Hint:** Position the box appropriately in the coordinate system and apply the Divergence Theorem along with Exercise 22.)
- Repeat Exercise 23 for the ball of radius  $R$ .

25. Repeat Exercise 23 for the right circular cone of radius  $R$  and height  $h$ .
26. If  $S$  is a surface such as in Exercise 22, use the Divergence Theorem to show that the volume of the solid  $D$  bounded by  $S$  can also be found as

$$V(D) = \iint_S x \, dy \, dz = \iint_S y \, dz \, dx = \iint_S z \, dx \, dy.$$

27. Let  $Q$  be a single point charge at the origin, as in our discussion that follows Example 2. Show that  $\nabla \cdot \mathbf{E} = 0$  for the electric field  $\mathbf{E}$  created by  $Q$ .
28. Mimic our discussion following Example 2 to prove the following, slightly more general version of Gauss' Law.

If  $S$  is a closed surface satisfying the hypotheses of the Divergence Theorem,  $k$  is a constant and  $\mathbf{F} = \frac{k\mathbf{r}}{|\mathbf{r}|^3}$  is an inverse square field, then the flux of  $\mathbf{F}$  over  $S$  is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \begin{cases} 4k\pi & \text{if } S \text{ encloses the origin.} \\ 0 & \text{otherwise} \end{cases}$$

- 29.\* Recall the uniformly charged sphere of radius  $R$  and total charge  $Q$  from Exercises 55–57 of Section 15.6. In this problem, you will provide a “second solution” to the aforementioned Exercise 57 as follows. Use Gauss' Law to show that the electric field  $\mathbf{E}$  due to the sphere is

$$\mathbf{E} = \begin{cases} \frac{\varepsilon Q \mathbf{e}_r}{r^2} & \text{if } R < r \\ 0 & \text{if } r < R. \end{cases}$$

(Recall  $\mathbf{e}_r$  from Exercise 54 of Section 15.1.)

**Hint:** Note that  $\mathbf{E}$  is a central force as in Exercise 29 of Section 15.7. Using the notation  $\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r$ , show that the flux across a sphere of radius  $r$  is  $4\pi r^2 E(r)$  and use Gauss' Law. Finally, note that if  $r < R$ , the total charge enclosed by any sphere of radius  $r$  is zero, and hence so is  $\mathbf{E}$ .)

30. Use the Divergence Theorem to provide a second solution to Exercise 28 of Section 15.7. (**Hint:** See Exercise 39 of Section 15.4.)
- 31.\* Prove that if  $\mathbf{F}$  has continuous partial derivatives, then its divergence at a point  $P$  can be obtained from the formula

$$\nabla \cdot \mathbf{F}(P) = \lim_{R \rightarrow 0} \frac{3}{4\pi R^3} \iint_{S_R(P)} \mathbf{F} \cdot \mathbf{n} \, d\sigma,$$

where  $S_R(P)$  is the sphere of radius  $R$  centered at  $P$ . (**Hint:** Use the Divergence Theorem; then argue that the divergence function “assumes its average value” somewhere in the closed ball bounded by  $S_R(P)$ . For a refresher on average value, see Section 5.2.)

**32–35** Prove the identity, assuming the scalar-valued functions  $f$  and  $g$  have continuous partial derivatives at least through the second order, and  $S$  and  $D$  satisfy the hypotheses of the Divergence Theorem. Note that  $\nabla^2 f$  stands for  $f_{xx} + f_{yy} + f_{zz}$ , i.e.,  $\nabla^2 f = 0$  means that  $f$  satisfies Laplace's equation (see Section 13.3). Recall also that  $D_{\mathbf{n}} f$  stands for the directional derivative of  $f$  in the direction of  $\mathbf{n}$ . (For a review of directional derivatives, see Section 13.5.)

$$32.* \iint_S D_{\mathbf{n}} f \, d\sigma = \iiint_D \nabla^2 f \, dV$$

$$33.* \iint_S f D_{\mathbf{n}} g \, d\sigma = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV$$

(This is called *Green's first identity*. **Hint:** See Exercise 44 in Section 15.4.)

$$34.* \iint_S f D_{\mathbf{n}} f \, d\sigma = \iiint_D |\nabla f|^2 \, dV, \text{ if } f \text{ satisfies Laplace's}$$

equation on  $D$ . (**Hint:** You may give a direct proof, or use Exercise 33.)

$$35.* \iint_S (f D_{\mathbf{n}} g - g D_{\mathbf{n}} f) \, d\sigma = \iiint_D (f \nabla^2 g - g \nabla^2 f) \, dV \text{ (This}$$

is called *Green's second identity*. **Hint:** Use Green's first identity from Exercise 33 twice.)

## Chapter 15

### Review Exercises

**1–4** Match the given planar vector field  $\mathbf{F}(x, y)$  with its graph (labeled A–D).

1.  $\mathbf{F}(x, y) = \langle 1, y \rangle$

2.  $\mathbf{F}(x, y) = \left\langle \frac{x}{2}, y \right\rangle$

3.  $\mathbf{F}(x, y) = \langle 2y, -x \rangle$

4.  $\mathbf{F}(x, y) = \left\langle -x, \frac{y}{\sqrt{|x, y|}} \right\rangle$