Assuming, as we have been, that the second partial derivatives of **F** are continuous in D, the equality of mixed partial derivatives means any one of the above statements implies $\nabla \times \mathbf{F} = \mathbf{0}$ throughout D. Incorporating the latest theorem, we can summarize the relationships between conservation and curl by

$$\begin{aligned} \mathbf{F} = \nabla f \quad &\Rightarrow \quad \nabla \times \mathbf{F} = \mathbf{0} \quad \text{if } D \text{ is an open connected region,} \\ \nabla \times \mathbf{F} = \mathbf{0} \quad &\Rightarrow \quad \mathbf{F} = \nabla f \quad \text{if } D \text{ is an open and simply connected region.} \end{aligned}$$

15.7 Exercises

1–4 Verify Stokes' Theorem by showing that the integrals $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ and $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$ are equal.

1. $\mathbf{F}(x, y, z) = \langle -2y, 2x, y + z \rangle$, where S is the upper unit hemisphere centered at the origin

2. $\mathbf{F}(x, y, z) = \langle y - z, x + z, -x + y \rangle$, where S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$

3. $\mathbf{F}(x, y, z) = \langle -y, x, 1 \rangle$ (as in Example 1), where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane

4. $\mathbf{F}(x, y, z) = \langle y, z^2, 2x \rangle$, where S is the portion of the paraboloid $z = x^2 + y^2$ between the xy-plane and the plane $z = 9$

5. Calculate $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$ for the vector field $\mathbf{F}(x, y, z) = \langle -2y, 2x, y + z \rangle$, on the upper semiellipsoid S: $x^2 + y^2 + 4z^2 = 1$, $z \geq 0$, and compare your answer to the solution of Exercise 1.

6. Verify by calculation that the field of normal vectors for the upper half of the ellipsoid given in Example 2 is $\mathbf{r}_x \times \mathbf{r}_y = \left\langle \dfrac{x}{3\sqrt{9 - x^2 - y^2}}, \dfrac{y}{3\sqrt{9 - x^2 - y^2}}, 1 \right\rangle$.

7–14 Use Stokes' Theorem to evaluate the indicated line integral.

7. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -2y, x^2, 3z^2 \rangle$ and C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + 2y + z = 4$, with positive orientation when viewed from above

8. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle z^2, x^2, y^2 \rangle$ and C is the intersection of the cylinder $x^2 + y^2 = 4x$ and the plane $z = 2x$, with positive orientation when viewed from above

9. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -3y, 2x, z^2 \rangle$ and C is the triangle with vertices $(4, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$, with positive orientation when viewed from above

10. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle zy^2, z^3, 9y - 2x \rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(3, 4, 1)$, and $(0, 0, 2)$, with positive orientation when viewed from above

11. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -3z, x, 2y \rangle$ and C is the boundary of the disk $x^2 + y^2 \leq 1$, $z = 1$, with positive orientation when viewed from above

12. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -3z, x, 2y \rangle$ and C is the intersection of the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$, with positive orientation when viewed from above (Compare your answer to the solution of Exercise 11.)

13. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle 2x^3, 4x + y^2, e^{z^2} \rangle$ and C is the intersection of the paraboloids $2z = x^2 + y^2$ and $z = 6 - x^2 - y^2$, with positive orientation when viewed from above

14. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F} = \langle -xz, 2z, x - y^2 \rangle$ and C is the intersection of $z = x^2 + y^2$ and $z = 2x + 3$, with negative orientation when viewed from above

15–22 Use Stokes' Theorem to evaluate the surface integral.

15. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle -y, x, xyz \rangle$ and S is the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, oriented with an upward-pointing unit normal vector field

16. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle 4z, -3x, 2y \rangle$ and S is the portion of the paraboloid $z = x^2 + y^2$ with $0 \leq z \leq 9$, oriented with a downward-pointing unit normal vector field

17. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle x - y^2, x^4, z^2 \rangle$ and S is the triangle determined by the first-octant portion of the plane $x + y + z = 1$, oriented with an upward-pointing unit normal vector field

18. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle 2y^2, -z, 4x \rangle$ and S is the triangle determined by the first-octant portion of the plane $5x + y + 2z = 10$, oriented with an upward-pointing unit normal vector field

19. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle -2yz, 3xz, 2z^3 \rangle$ and S is the cone frustum $z = \sqrt{x^2 + y^2}$, $1 \le z \le 2$, oriented with an inward-pointing unit normal vector field

20. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle yz, -2xz, x^2 y \rangle$ and S is the portion of the cylinder $x^2 + y^2 = 4$, $2 \le z \le 4$, oriented with an outward-pointing unit normal vector field

21. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle y \sin(z^2), xy^2, xz + y \rangle$ and S is the portion of the paraboloid $x = y^2 + z^2$, $0 \le x \le 4$, oriented with an inward-pointing unit normal vector field

22. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle 3yx^2, x \ln(z^4), y \cos(xz^2) - 2x \rangle$ and S is the hemisphere $y = \sqrt{4 - x^2 - z^2}$, oriented with an inward-pointing unit normal vector field

23. Suppose, as in Exercise 3 of Section 15.6, that the (piecewise smooth) surface S is the graph of a function $z = g(x, y)$, defined on a domain R, and that S has a piecewise smooth boundary C. Prove that in this case, Stokes' Theorem takes the following form.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \nabla \times \mathbf{F} \cdot \langle -g_x, -g_y, 1 \rangle\, dA$$

24–26 Assume that both f and g have continuous second-order partial derivatives, and that both the surface S and its boundary C meet the conditions of Stokes' Theorem. Verify the statement by using Stokes' Theorem.

24. $\oint_C (f \nabla f) \cdot \mathbf{T}\, ds = 0$

25. $\oint_C (f \nabla g) \cdot \mathbf{T}\, ds = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma$

26. $\oint_C (f \nabla g + g \nabla f) \cdot \mathbf{T}\, ds = 0$

27. Let S be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 1$. If $\mathbf{F}$ is a vector field with continuous partials in an open region containing S, show that

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

28.* Prove the following generalization of Exercise 27. If the vector field $\mathbf{F}$ and the closed surface S satisfy the conditions of Stokes' Theorem, then

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

29.* The force field $\mathbf{F}$ is called a *central force* if it points directly away from, or toward, a point called the *center*, and its magnitude depends only on the distance from the center. In addition, we assume that this dependency is continuously differentiable, or in symbols, $\mathbf{F} = f(r)\mathbf{r}$, where the single-variable function f is continuously differentiable everywhere except possibly at zero. Show that if such a force is moving an object around a closed path that doesn't enclose the origin, then the total work done by the force is zero. (**Hint:** Show that the curl of the force field is zero.)

30. Consider the vector field

$$\mathbf{F} = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right\rangle$$

and show that while $\nabla \times \mathbf{F} = \mathbf{0}$, $\oint_C \mathbf{F} \cdot d\mathbf{r} \ne 0$ on a circle in the xy-plane centered at the origin. Does this contradict the theorem stating that $\nabla \times \mathbf{F} = \mathbf{0}$ implies $\mathbf{F}$ is conservative?

15.7 Technology Exercises

31. Write a program on a computer algebra system that accepts a vector field, the parametrizations of a surface S and its boundary C, and returns both integrals $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ and $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$. Use it to check your answers to Exercises 1–4.