where κ is the thermal conductivity constant for the substance of the ball. From Example 4, we know that the outward unit normal vector **n** at the point (x, y, z) on the surface $x^2 + y^2 + z^2 = r^2$ is

$$\mathbf{n}(x, y, z) = \frac{1}{r}\langle x, y, z\rangle,$$

so on S,

$$\mathbf{F}\cdot\mathbf{n} = \frac{2\kappa}{r}\langle x, y, z\rangle\cdot\langle x, y, z\rangle = \frac{2\kappa}{r}\left(x^2 + y^2 + z^2\right) = \frac{2\kappa}{r}\left(r^2\right) = 2\kappa r.$$

Hence, the flow of heat per unit time across S is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 2\kappa r\iint_S d\sigma = 2\kappa r\left(4\pi r^2\right) = 8\kappa\pi r^3.$$

15.6 Exercises

1. Use the substitution suggested in Example 1 to conclude that

$$M = \iint_S \rho\,d\sigma = \frac{\pi\left(25\sqrt{5} - 11\right)}{60}.$$

2. Use the same substitution as in Exercise 1 to finish Example 2 by showing that

$$\bar{z} = \frac{M_{xy}}{M} = \frac{2\left(25\sqrt{5} + 4\right)}{7\left(25\sqrt{5} - 11\right)} \approx 0.38.$$

3. Suppose the surface S is given as the graph of a function $z = g(x, y)$, defined on a domain R. Prove that for a continuous function $f = f(x, y, z)$ defined on S, the surface integral of f over S is

$$\iint_S f\,d\sigma = \iint_R f\left(x, y, g(x, y)\right)\sqrt{1 + \left[g_x\right]^2 + \left[g_y\right]^2}\,dA.$$

(**Hint:** Parametrize S by using x and y as parameters.)

4–25 Parametrize the surface S and evaluate the indicated surface integral. (You may use the formula from Exercise 3 whenever feasible. Use polar or spherical coordinates where needed.)

4. $\iint_S 2z\,d\sigma$, where S is the first-octant portion of the plane $x + 2y + z = 4$

5. $\iint_S y\,d\sigma$, where S is the graph of $2x + y + z = 6$ above the square $[0,2]\times[0,2]$

6. $\iint_S 3x\,d\sigma$, where S is the graph of $x^2 + z = 9$ above the rectangle $[0,3]\times[0,4]$

7. $\iint_S x\,d\sigma$, where S is the first-octant portion of the plane $x + y + z = a$ $(a > 0)$

8. $\iint_S z^2\,d\sigma$, where S is the intersection of the plane $2x + 2y + z = 0$ and the interior of the cylinder $x^2 + y^2 = 1$

9. $\iint_S \left(x^2 + y^2 + z\right)d\sigma$, where S is the graph of $x + 2y + z = 2$ above the rectangle $[0,1]\times[0,2]$

10. $\iint_S 9z\,d\sigma$, where S is the surface $z = y^3$ over the rectangle $[-1,1]\times[0,1]$

11. $\iint_S (2xy + z)\,d\sigma$, where S is the graph of $2y - x + z = 4$ above the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$

12. $\iint_S (x + y + z)\,d\sigma$, where S is the graph of $z = \sqrt{1 - x^2}$ above the square $[0,1]\times[0,1]$

13. $\iint_S (z - y)\,d\sigma$, where S is the graph of $z = 2x^2 + y$ above the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$

14. $\iint_S 2z\,d\sigma$, where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane

15. $\iint_S \left(z + 2x^2\right)d\sigma$, where S is the surface $z = y^2 - x^2$ above the half disk bounded by $y = \sqrt{1 - x^2}$ and the x-axis

16. $\iint_S y^2\,d\sigma$, where S is the upper unit hemisphere $z=\sqrt{1-y^2-x^2}$

17. $\iint_S \left(x^2+y^2\right)d\sigma$, where S is the portion of the paraboloid $z = 2 - x^2 - y^2$ above the xy-plane

18. $\iint_S z\,d\sigma$, where S is the portion of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 1$ and $z = 3$

19. $\iint_S \frac{y^2}{1-z}\,d\sigma$, where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ with $\frac{3}{4} \le z \le 1$

20. $\iint_S \sin z\,d\sigma$, where S is the cylinder $x^2+y^2=1,\ 0\le z\le \pi/2$

21. $\iint_S \left(2x^2+2y^2+2z^2\right)d\sigma$, where S is the unit sphere centered at the origin; use spherical coordinates

22. $\iint_S \left(yz+y^3\right)d\sigma$, where S is the box $[0,1]\times[0,2]\times[0,3]$

23. $\iint_S \left(x^2+2xz\right)d\sigma$, where S is the tetrahedron formed by the plane $x + 2y + 2z = 4$ and the coordinate planes

24. $\iint_S z\,d\sigma$, where S is the solid bounded by the cylinder $x^2 + y^2 = 1$, the xy-plane, and the plane $2z = x + 3$

25. $\iint_S xyz\,d\sigma$, where S is the cone frustum $z=3-\sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$

26. Find the centroid of the surface in Example 1, assuming constant density.

27. Determine the mass M of the half cylinder S defined by $\mathbf{r}(s,t)=\langle 2\cos s, 2\sin s, t\rangle$, $0\le s\le \pi$, $0\le t\le 2$, if its mass density is $\rho(x,y,z)=y+z$.

28. Determine the mass M of the cone S defined by $\mathbf{r}(s,t)=\langle s\cos t, s\sin t, s\rangle$, $0\le s\le 1$, $0\le t\le 2\pi$, if its mass density is proportional to the distance from the z-axis.

29. Find the center of mass of the half cylinder given in Exercise 27.

30. Find the center of mass of the cone surface given in Exercise 28.

31. Determine the mass of the portion of the thin spherical shell $z=\sqrt{25-x^2-y^2}$ between the planes $z = 3$ and $z = 4$ if its density at any point is proportional to the distance from the xy-plane. (**Hint:** Consider spherical coordinates.)

32. Find the centroid of the thin hemispherical surface $g(x,y)=\sqrt{R^2-x^2-y^2}$ if it has constant density ρ. (See the hint given in Exercise 31.)

33. Determine the second moments about the coordinate axes of the sphere $x^2 + y^2 + z^2 = R^2$ if it has constant density ρ. (See the hint given in Exercise 31.)

34–52 Determine the indicated flux of the vector field **F** across the surface S. Unless otherwise specified, the surfaces are oriented with outward-pointing normal vectors.

34. The flux of $\mathbf{F}=\langle 0,0,c\rangle$ across the hemisphere given in Exercise 32

35. The flux of $\mathbf{F}=\langle x,y,2\rangle$ out of the solid R bounded by $3z = x^2 + y^2$ and the plane $z = 3$

36. The flux of $\mathbf{F}=\langle 2,z,y\rangle$ across the first-octant portion of the unit hemisphere centered at the origin

37. The flux of $\mathbf{F}=\langle -y,x,2\rangle$ across the first-octant portion of the unit hemisphere centered at the origin

38. The flux of $\mathbf{F}=\langle x,y,z\rangle$ across the first-octant portion of the unit hemisphere centered at the origin

39. The flux of $\mathbf{F}=\langle x,y,z\rangle$ across the first-octant portion of the cylinder $x^2 + y^2 = 1$, between the planes $z = 0$ and $z = 1$

40. The flux of $\mathbf{F}=\langle x^2,xy,xz\rangle$ across the first-octant portion of the hemisphere of radius R, centered at the origin

41. The flux of $\mathbf{F}=\langle 0,0,x^2+y^2\rangle$ across the paraboloid $z = x^2 + y^2 + 1$, $1 \le z \le 10$

42. The flux of $\mathbf{F}=\langle 0,y^2,x^2+z^2\rangle$ across the hemisphere $z=\sqrt{4-x^2-y^2}$

43. The flux of $\mathbf{F}=\langle y,-x,6\rangle$ across the portion of the upper unit hemisphere centered at the origin that projects onto the disk $x^2+y^2\le\frac{1}{2}$ (Note that this can be interpreted as the upward flux through the hemisphere of a fluid with a “rotating flow.”)

44. The flux of the vector field given in Exercise 43 across the portion of the paraboloid $z = 1 - x^2 - y^2$ that projects onto the disk $x^2 + y^2 \le \frac{1}{2}$ (Compare your answer with the solution of Exercise 43.)

45. The flux of the vector field given in Exercise 43 across the portion of the cone $z = 1 - \sqrt{x^2 + y^2}$ that projects onto the disk $x^2 + y^2 \le \frac{1}{2}$ (Compare your answer with the solution of Exercise 43 or 44.)

46. The flux of $\mathbf{F} = \langle -2y, x, 2z \rangle$ out of the solid R bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 6$

47. The flux of $\mathbf{F} = \langle x, y, 2 \rangle$ out of the solid R bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

48. The flux of $\mathbf{F} = \langle 2x, y, z \rangle$ across the portion of the surface $z = 1 - x^2 - y^2$ above the xy-plane

49. The flux of $\mathbf{F} = \langle -y, x, z^2/2 \rangle$ across the cone frustum $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 3$

50. The flux of $\mathbf{F} = \langle x^2, 2y, yz \rangle$ out of the cube in Example 3

51. The flux of $\mathbf{F} = \langle 0, y, 2z \rangle$ out of the solid region bounded above by $z = 4 - x^2 - y^2$ and below by the plane $z = 2$

52. The flux of $\mathbf{F} = \langle x, y, 1 \rangle$ out of the solid bounded by the paraboloid $z = x^2 + y^2$ and the planes $z = 1$ and $z = 4$

53. Solve Example 6 if the temperature is inversely proportional to the square of the distance from the origin.

54. Suppose the temperature function of the solid ball $x^2 + y^2 + z^2 \le 4$ is $T(x, y, z) = 70 - 0.1x^2$ and $\kappa = 3$. Determine the heat flow out of the region.

55–57 A discussion analogous to the one given preceding Exercises 50–53 in Section 15.2 yields the formula below. If S is a thin surface with electrical charge density $q(x,y,z)$, the electrostatic potential at a point P away from the surface is obtained from the surface integral

$$V(P) = \varepsilon \iint_S \frac{q(x,y,z)}{r_P(x,y,z)}\, d\sigma,$$

where $r_P(x,y,z)$ is the distance between (x,y,z) and P.

Use the above formula in Exercises 55–57.

55.* Suppose a uniformly charged sphere of radius R and total charge Q is centered at the origin. If point P is r units from the center of the sphere, then show that

$$V(P) = \varepsilon \int_0^{\pi} \int_0^{2\pi} \frac{Q \sin\varphi}{4\pi\sqrt{R^2 + r^2 - 2rR\cos\varphi}}\, d\theta\, d\varphi.$$

(**Hint:** Because of uniform charge distribution, the charge density on the sphere is given by $q(x,y,z) = \frac{Q}{4\pi R^2}$. Notice also that because of radial symmetry, it suffices to pick P on one of the coordinate axes.)

56.* Use Exercise 55 to show that

$$V(P) = \frac{\varepsilon Q}{2rR}\left(|R + r| - |R - r|\right).$$

(**Hint:** Substitute $R^2 + r^2 - 2rR\cos\varphi = u$.)

57.* Use Exercises 55 and 56 to conclude that the electrostatic potential for a uniformly charged sphere r units from its center is as follows.

$$V(P) = \begin{cases} \varepsilon \dfrac{Q}{r} & \text{if } P \text{ is outside the sphere} \\ \varepsilon \dfrac{Q}{R} & \text{if } P \text{ is inside the sphere} \end{cases}$$

Note what this means is that the potential is constant inside the sphere, while outside the sphere the potential function behaves as if all of the charge Q were concentrated at the origin!

15.6 Technology Exercises

58–61 Use a graphing utility.

58. Determine the center of mass of the hemispherical surface given in Exercise 31.

59. Determine the moments of inertia of the surface given in Exercise 31.

60. Determine the mass and centroid of the thin parabolic surface $z = 1 - x^2 - y^2$, $z \ge 0$, if it has constant density ρ.

61. Find the second moments about the coordinate axes of the surface given in Exercise 58.