```
In[6]:= ParametricPlot3D[{x, y, z}, {r, 0, 1.38}, {θ, 0, 2 Pi},
         PlotPoints → 100, Mesh → None, Boxed → False,
         Axes → False, PlotRange → All, ViewPoint → {1, 1, 10}]
Out[6]=
```

Figure 11b

15.5 Exercises

1–4 Describe the surface with a vector function of two parameters. (Answers will vary.)

1. The graph of $y = \frac{1}{3}x$, $0 \le x \le 6$, revolved about the x-axis

2. The graph of $z = 1/x$, $0 \le x \le 8$, revolved about the x-axis

3. The graph of $x = 1 - z^2$, $-1 \le z \le 1$, revolved about the z-axis

4. The graph of $z = \cos y$, $-\pi/2 \le y \le \pi/2$, revolved about the y-axis

5–13 Match the parametric surface with its graph (labeled A–I).

5. $\mathbf{r}(s,t) = \langle 3\cos s, 3\sin s, t\rangle$, $0 \le s \le 2\pi$, $0 \le t \le 8$

6. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 2s\rangle$, $0 \le s \le 5$, $0 \le t \le 2\pi$

7. $\mathbf{r}(s,t) = \langle \cos 2t \sin s, \cos s \cos 2t, \sin t\rangle$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$

8. $\mathbf{r}(s,t) = \left\langle s, \frac{\cos t}{s^2}, \frac{\sin t}{s^2}\right\rangle$, $\frac{1}{3} \le s \le 3$, $0 \le t \le 2\pi$

9. $\mathbf{r}(s,t) = \langle 2t - s, 1 + 2s + t, 1 + s - 3t\rangle$, $0 \le s \le 5$, $0 \le t \le 5$

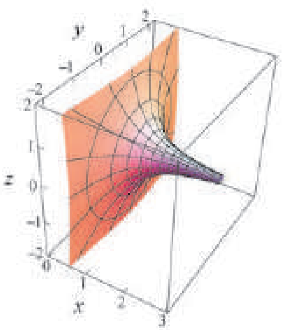
10. $\mathbf{r}(s,t) = \langle s\sin t, s\cos t, \cos s\rangle$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$

11. $\mathbf{r}(s,t) = \langle \sin s \cos t, \sin s \sin t, \cos s\rangle$, $0 \le s \le \pi/2$, $0 \le t \le 2\pi$

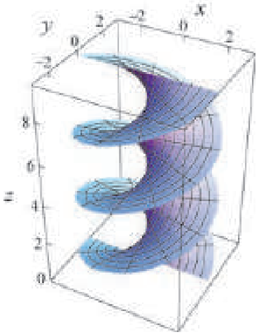
12. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, t\rangle$, $-3 \le s \le 3$, $0 \le t \le 3\pi$

13. $\mathbf{r}(s,t) = \langle \cos s \sin t, 2\sin s \sin t, \cos t\rangle$, $0 \le s \le \pi$, $0 \le t \le 2\pi$

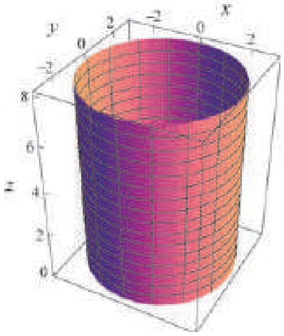
A.



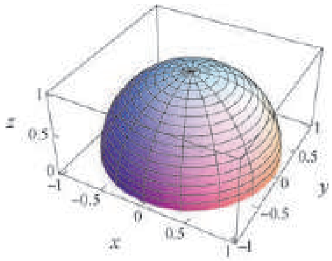
B.



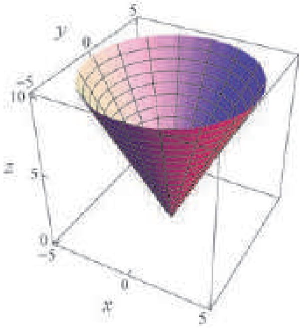
C.



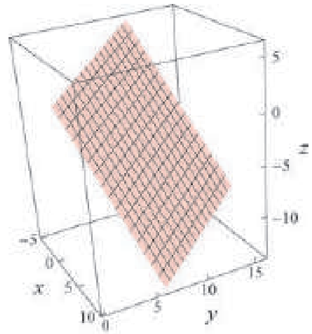
D.



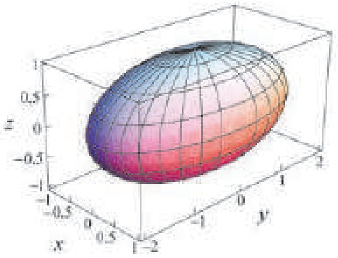
E.



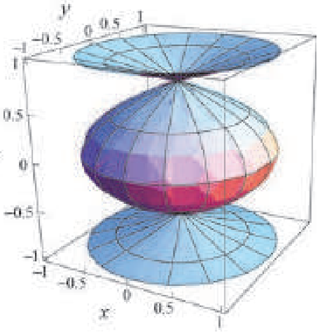
F.



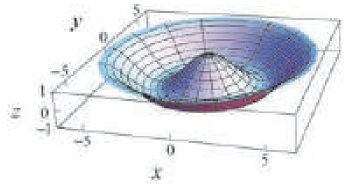
G.



H.



I.



14–20 Identify the surface by examining its grid curves.

14. $\mathbf{r}(s,t)=\langle t,\cos s,\sin s\rangle,\quad 0\le s\le 2\pi,\quad -\infty<t<\infty$

15. $\mathbf{r}(s,t)=\langle R\sin s\cos t, R\sin s\sin t, R\cos s\rangle,$
$0\le s\le \pi/2,\quad 0\le t\le \pi/2$

16. $\mathbf{r}(s,t)=\langle s\cos t, s\sin t, s\rangle,$
$-\infty<s<\infty,\quad 0\le t\le 2\pi$

17. $\mathbf{r}(s,t)=\langle s,t,2s^2+t^2\rangle,$
$-\infty<s<\infty,\quad -\infty<t<\infty$

18. $\mathbf{r}(s,t)=\langle (3+\cos t)\cos s,(3+\cos t)\sin s,\sin t\rangle,$
$0\le s\le 2\pi,\quad 0\le t\le 2\pi$

19. $\mathbf{r}(s,t)=\langle s\cos t, s\sin t, 4-s^2\rangle,$
$0\le s\le 2,\quad 0\le t\le 2\pi$

20. $\mathbf{r}(s,t)=\left\langle s,\frac{\cos t}{s},\frac{\sin t}{s}\right\rangle,\quad \frac{1}{4}\le s\le 4,\quad 0\le t\le 2\pi$

21–31 Obtain a parametrization for the indicated surface. (Answers will vary.)

21. $z=x+y$

22. $z=xy$

23. $x^2+y^2=1,\ 0\le z\le 1$

24. $z=\sqrt{x^2+y^2},\ z\le 2$

25. $x^2+y^2+z^2=4$

26. $z=x^2+y^2$

27. $z=-\sqrt{1-x^2-y^2}$

28. $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$

29. $x^2+4y^2=4$

30. The intersection of $z=x^2+y^2$ with the interior of $x^2+y^2=4$

31. The portion of the sphere $x^2+y^2+z^2=4$ outside the double cone $z^2=3x^2+3y^2$

32–37 Construct an equation for the plane tangent to the surface at the indicated point.

32. $\mathbf{r}(s,t)=\langle s^2,s+t,t^2\rangle;\quad \mathbf{r}(1,1)$

33. $\mathbf{r}(s,t)=\langle s,t,2s^2+t^2\rangle;\quad \mathbf{r}(1,2)$

34. $\mathbf{r}(s,t)=\langle 2st,s^2,t^2\rangle;\quad \mathbf{r}(1,-3)$

35. $\mathbf{r}(s,t)=\langle 2s\cos t, s\sin t, s^2\rangle;\quad \mathbf{r}\left(2,\frac{\pi}{4}\right)$

36. $\mathbf{r}(s,t)=\left\langle 2s^2,st^2,\frac{st}{2}\right\rangle;\quad \mathbf{r}(1,2)$

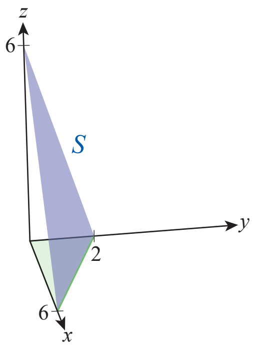
37. $\mathbf{r}(s,t)=\langle s\sin t, s^4, s\cos t\rangle;\quad \mathbf{r}\left(1,\frac{\pi}{3}\right)$

38. Parametrize the sphere of radius R as in Example 2, and show that its normal vector $\mathbf{n}(\theta,\varphi)$ is a constant multiple of $\mathbf{e}_r$ (see Exercise 54 of Section 15.1). What is that constant?

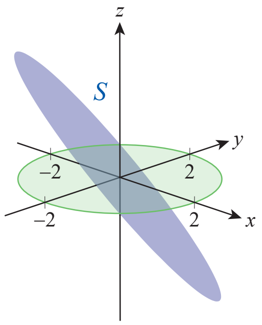
39. Verify that the parametric approach to surface area is consistent with the surface area of a solid of revolution discussed in Section 6.3. (**Hint:** Let $f(x) \geq 0$ be a continuously differentiable single-variable function defined on $[a,b]$, and rotate its graph around the x-axis. Parametrize the resulting surface of revolution as in Example 1. Calculate $\mathbf{n} = \mathbf{r}_s \times \mathbf{r}_t$ as in Example 5 and find the surface area $A = \int_0^{2\pi}\int_a^b |\mathbf{n}|\, dA$ after showing that $|\mathbf{n}| = |\mathbf{r}_s \times \mathbf{r}_t| = f(x)\sqrt{1+[f'(x)]^2}$.)

40–56 Find the area of the surface S. (Use polar coordinates wherever they simplify your calculations.)

40. S is the first-octant portion of the plane $x + 3y + z = 6$.

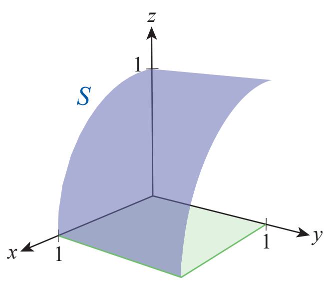


41. S is the intersection of the plane $2x + 2y + z = 0$ and the interior of the cylinder $x^2 + y^2 = 4$.



42. S is the graph of $z = 3 + 2x - y$ defined on the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

43. S is the graph of $z = \sqrt{1-x^2}$ defined on the square $[0,1]\times[0,1]$.



44. S is the graph of $z = 2x^2 + y$ defined on the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

45. S is the portion of the paraboloid $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 1-s^2\rangle$ above the xy-plane.

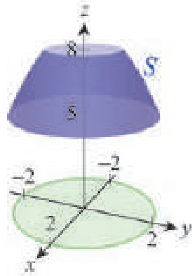
46. S is the surface of the cone $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, s\rangle$ between the planes $z = 1$ and $z = 2$.

47. S is the portion of the paraboloid $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, s^2\rangle$ between the planes $z = 4$ and $z = 9$.

48. S is the portion of the cone $z = \sqrt{x^2+y^2}$ defined on the triangle with vertices $(0,0)$, $(1,1)$, and $(0,1)$.

49. S is the graph of $z = y^2 - x^2$ defined on the first-quadrant portion of the disk of radius 2 centered at the origin.

50. S is the surface $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, 9-s^2\rangle$ between the planes $z = 5$ and $z = 8$.



51. S is the portion of the paraboloid $\mathbf{r}_1(s,t) = \langle s\cos t, s\sin t, s^2/2\rangle$ between the cylinders $\mathbf{r}_2(s,t) = \langle \cos s, \sin s, t\rangle$ and $\mathbf{r}_3(s,t) = \langle 2\cos s, 2\sin s, t\rangle$.

52. S is the portion of $x^2 + y^2 = z^2 + 1$ between the xy-plane and $z = \sqrt{3}$.

53. S is the first-quadrant portion of $z + y^2 = 1$, $z \geq 0$, between the yz-plane and $x = 1$.

54. S is the torus $\mathbf{r}(s,t) = \langle (2+\cos t)\cos s, (2+\cos t)\sin s, \sin t\rangle$, $0 \leq s \leq 2\pi$, $0 \leq t \leq 2\pi$.

55. S is the portion of the hemisphere $z = \sqrt{25-x^2-y^2}$ between the planes $z = 3$ and $z = 4$.

56. S is the portion of the cylinder $x^2 + y^2 = 25$ between the planes $z = 3$ and $z = 4$. (Compare your answer to the solution of Exercise 55.)

57.* Generalize Exercise 54 to arrive at the formula for the surface area of the torus parametrized as $\mathbf{r}(s,t)=\langle(a+b\cos t)\cos s,(a+b\cos t)\sin s,b\sin t\rangle$, where $a>b>0$. (**Hint:** Consider the circle of radius b in the xz-plane, centered at $(a,0,0)$, and rotate it around the z-axis.)

58.* Generalize your observations made in Exercises 55 and 56 to prove Archimedes' famous result:

The surface area of the section of the sphere $x^2+y^2+z^2=R^2$ between the planes $z=a$ and $z=b$ equals the surface area of the corresponding section of the circumscribed cylinder $x^2+y^2=R^2$.

59.* Show that surface area is independent of parametrization; that is, prove the following statement.

Let R_1 and R_2 be regions in the plane enclosed by simple closed paths, and let $\mathbf{r}_1 : R_1 \to \mathbb{R}^3$, $\mathbf{r}_2 : R_2 \to \mathbb{R}^3$ be continuously differentiable, one–to–one parametrizations of the same surface, that is, $\mathbf{r}_1(R_1)=\mathbf{r}_2(R_2)$. With the usual notation $\mathbf{n}_i=(\mathbf{r}_i)_s\times(\mathbf{r}_i)_t$, $i=1,2$, prove that

$$\iint_{R_1}|\mathbf{n}_1|\,dA=\iint_{R_2}|\mathbf{n}_2|\,dA.$$

(**Hint:** Use the rules for differentiation of inverses and change of variables.)

Concept Check

60–63 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

60. All grid curves of a parametric surface are parallel to a coordinate plane.

61. Before we can evaluate its area, a surface S must be parametrized.

62. A parametric surface $\mathbf{r}(s,t)$ has a tangent plane at any point of smoothness $\mathbf{r}(s_0,t_0)$.

63. The area of a parametric surface is a limit of Riemann sums.

15.5 Technology Exercises

64. Describe in words the grid curves of the parametric surface given in Exercise 12; then use a graphing utility to sketch the surface. Finally, use your technology to approximate its area. (**Note:** This is an example of what are called *helicoid surfaces*.)

65. Sketch the parametric surface

$$\mathbf{r}(s,t)=\left\langle\left(\sqrt{25-t^2}-3\right)\cos s,\left(\sqrt{25-t^2}-3\right)\sin s,t\right\rangle,$$
$$0\le s\le 2\pi,\quad -4\le t\le 4,$$

using a graphing utility and verify that the graph resembles a football. Change the coefficients to make the "football" appear "thinner" or "thicker," respectively. (Carefully determine the domain for each parameter. Answers will vary.)

66. The parametric surface

$$\mathbf{r}(s,t)=\left\langle\cos t+s\cos\frac{t}{2},3\sin t+s\cos\frac{t}{2},s\sin\frac{t}{2}\right\rangle,$$
$$-\frac{1}{2}\le s\le\frac{1}{2},\quad 0\le t\le 2\pi$$

is an example of the famous *Möbius strip* (after the German mathematician August Ferdinand Möbius). Use a graphing utility to graph and examine this surface. Can you think of a way to produce such a surface using a strip of paper and tape? Notice that if you start sliding your finger along one side of the surface, you will eventually arrive back at your starting point without crossing any edges! Surfaces with this property are called *nonorientable*, or *one-sided*.

67. Graph and examine the surface

$$\mathbf{r}(s,t)=\langle(a+\sin t)\cos s,(a+\sin t)\sin s,t\rangle,$$
$$0\le s\le 2\pi,\quad 0\le t\le 2\pi$$

for several values of the parameter a. Then use your technology to find its surface area if $a=2$.