

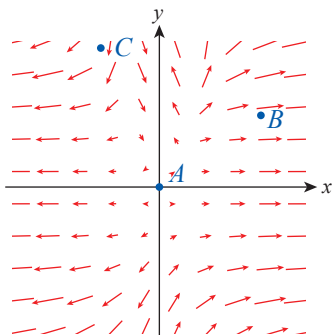
15.4 Exercises

1–10 Find the divergence of the vector field \mathbf{F} and evaluate the divergence at the given points.

- $\mathbf{F}(x, y) = \langle 5x^3 - 2xy, 4xy^2 \rangle$; $A(2, 1)$, $B(1, -3)$
- $\mathbf{F}(x, y) = \langle xe^{-y}, y^2 \cos x \rangle$; $A(0, -1)$, $B(\pi/2, 2/\pi)$
- $\mathbf{F}(x, y, z) = \langle x^2, 3xz^2, -2yz \rangle$; $A(2, -1, 0)$,
 $B(1, 1, 4)$
- $\mathbf{F}(x, y, z) = \left\langle \frac{x^2}{2}, \frac{y^2}{2}, \frac{z^2}{2} \right\rangle$; $A(-1, 3, 0)$,
 $B(5, -7, 2)$
- $\mathbf{F}(x, y, z) = \langle 2yz, 2xz, 2xy \rangle$; $A(1, 1, 1)$,
 $B(2, -3, 4)$
- $\mathbf{F}(x, y, z) = \langle xy^2, xz + 2y, z^2 \rangle$; $A(2, 0, -2)$,
 $B(4, 9, 0)$
- $\mathbf{F}(x, y, z) = \langle x + y^3, e^y, z^4 \rangle$; $A(1, 0, -3)$, $B(2, 1, 4)$
- $\mathbf{F}(x, y, z) = \langle \cos(xy^2), x, 2y - z \rangle$; $A(-2, 0, 1)$,
 $B(\pi/2, 5, 5)$
- $\mathbf{F}(x, y, z) = \langle e^{xyz}, e^{yz}, e^{xz} \rangle$; $A(1, 0, 1)$, $B(e, 1, -1)$
- $\mathbf{F}(x, y, z) = \left\langle x^2y, xy - z^2, \frac{2z^2}{x^2 + y^2} \right\rangle$; $A(0, 1, 3)$,
 $B\left(-1, 1, \frac{1}{2}\right)$

11–20. Find the curl of each vector field given in Exercises 1–10.

21. Use the figure to decide whether the divergence of the vector field at the given points is positive, negative, or zero.



22. You may have noticed by now that for a vector field $\mathbf{F} = \langle P, Q, R \rangle$, those terms in P involving only x do not affect the curl; neither do those terms in Q involving only y . A similar statement holds for R and z . For example,

$$\text{curl} \left\langle x^2 \cos x, y^4 + z, \frac{1}{\sqrt{1+z^2}} + y^2 \right\rangle = \text{curl} \langle 0, z, y^2 \rangle.$$

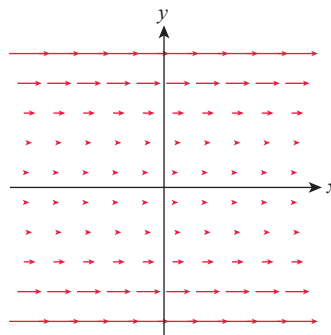
Prove the above assertions.

23–24 Use the “shortcut” suggested by Exercise 22 to evaluate the indicated curl.

$$23. \text{curl} \langle y + \ln(x^2 + 1), y^3 - \sin y, e^{\sin z} + 1 \rangle$$

$$24. \text{curl} \left\langle z - \ln \sqrt{1+x^2}, x^2 + \arctan(y^2), \frac{z}{1+z^2} \right\rangle$$

25. Determine the curl of $\mathbf{F}(x, y) = \langle y^2, 0 \rangle$ and provide a physical interpretation of the result (as done in Example 2). See the figure below for guidance.



26–29 Find $\text{curl}(\text{curl } \mathbf{F})$ for the given vector field.

$$26. \mathbf{F}(x, y, z) = \langle x, -y, xyz \rangle$$

$$27. \mathbf{F}(x, y, z) = \langle xy^2, xz, -4xz^2 \rangle$$

28. The vector field given in Exercise 3

29. The vector field given in Exercise 6

30–37 Decide whether or not the expression has meaning. If an expression does have meaning, state whether it is a vector field or a scalar field.

$$30. \nabla \times (\nabla \cdot \mathbf{F}) \qquad 31. \nabla \cdot (\nabla \cdot \mathbf{F})$$

$$32. \nabla \cdot (\nabla \times \mathbf{F}) \qquad 33. \nabla (\nabla \cdot \mathbf{F})$$

$$34. \nabla (\nabla \times \mathbf{F}) \qquad 35. \nabla \cdot (\nabla f)$$

$$36. \nabla \times (\nabla f) \qquad 37. \nabla (\nabla f)$$

38–47 Supposing that all of the appropriate partial derivatives of the vector fields \mathbf{F} and \mathbf{G} are continuous, prove the given statement. (Assume that f and g are at least twice differentiable scalar fields; a , b denote constants.)

38. $\nabla \times (\nabla f) = \mathbf{0}$
39. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
40. $\nabla \cdot (a\mathbf{F} + b\mathbf{G}) = a(\nabla \cdot \mathbf{F}) + b(\nabla \cdot \mathbf{G})$
41. $\nabla \times (a\mathbf{F} + b\mathbf{G}) = a(\nabla \times \mathbf{F}) + b(\nabla \times \mathbf{G})$
42. $\nabla \times (\nabla f + \nabla \times \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F})$
43. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
44. $\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + (\nabla f) \cdot \mathbf{F}$
45. $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$
46. $\nabla \cdot (\nabla f \times \nabla g) = 0$
47. $\nabla \cdot (\nabla fg) = f \nabla \cdot (\nabla g) + g \nabla \cdot (\nabla f) + 2(\nabla f) \cdot (\nabla g)$

48–51 Verify Green's Theorem by demonstrating the equality

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA, \text{ where } P(x, y) = xy^2,$$

$Q(x, y) = 2x$, and C is a smooth closed curve enclosing the region R as specified.

48. R : The triangle with vertices $(0,0)$, $(3,0)$, and $(0,3)$
49. R : The square $[-1,1] \times [-1,1]$
50. R : The region bounded by the graphs of $y = x^2$ and $y = x$
51. R : The unit circle (disk) centered at the origin
52. Parametrize curve C given in Example 4 and calculate the line integral of \mathbf{F} around C to show $\oint_C y^2 dx + 3xy dy = \frac{1}{3}$.

53–62 Use Green's Theorem to evaluate the line integral. (Pay attention to the orientation of the path as indicated by the integration symbol.)

53. $\oint_C xy dx + (2x + y) dy$, where C is the boundary of the region between the graphs of $y = 1 - x^2$ and the x -axis, oriented counterclockwise
54. $\oint_C 4xy dx + \ln(x^2 + y^2) dy$, where C is the cardioid $r = 1 + \cos \theta$, oriented counterclockwise

55. $\oint_C 2y^2 dx - xy dy$, where C is the graph of $y = \sqrt{x}$ from the origin to $(1,1)$, followed by $y = x^2$ from $(1,1)$ back to $(0,0)$

56. $\oint_C x^2 y dx + (x^3 + y^2) dy$, where C is the boundary of the region bounded by the x -axis, the graphs of $y = x^3$ and $x = 2$, oriented counterclockwise

57. $\oint_C 4x^2 y^3 dx - x^3 y dy$, where C is the boundary of the region bounded by the coordinate axes and the line $y = 2 - 2x$, oriented clockwise

58. $\oint_C [\cos(x^2) - 2y] dx + [4x + \ln(y^4)] dy$, where C is the unit circle centered at the origin, with positive orientation

59. $\oint_C (y^3 - \sqrt{x^2 + 2}) dx + (3xy^2 - \cos y) dy$, where C is defined as the border of a map of Louisiana contained in the first quadrant (Assume that the border is a smooth, simple, closed curve.)

60. $\oint_C e^x dx + y^2 e^x dy$, where C is the triangle with vertices $(0,0)$, $(1,2)$, and $(5,0)$, oriented counterclockwise

61. $\oint_C [y + 2 \tan^{-1}(x/4)] dx - (e^{x^2+1} + 2 \sec y) dy$, where C is the boundary of the region between the graphs of $y = |x|$ and $y = 2 - |x|$, with positive orientation

62. $\oint_C x e^y dx + xy dy$, where C is the boundary of the region between the graphs of $x = y^2$ and $x - y = 2$, with clockwise orientation

63–67 In these exercises, we will revisit some exercises from Section 15.2. Use Green's Theorem to provide a second solution and verify that your answers agree.

63. Determine $\int_C (x^2 y + xy^2) dx + x^3 dy$, where C is the unit circle centered at the origin. (See Exercise 27 of Section 15.2.)

64. Determine $\int_C y dx + xy dy$, where C is the parabola $y = x^2$ joining the origin with $(1,1)$, followed by the line segment from $(1,1)$ back to the origin. (See Exercise 28 of Section 15.2.)

65. Find the line integral $\int_C 2xy dx + x^2 dy$ on the curve given in Exercise 64. (See Exercise 29 of Section 15.2.)

66. Find $\int_C (x^2 + y) dx + xy dy$ along the closed path $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from the origin to $(1,0)$, C_2 is the line segment from $(1,0)$ to $(2,1)$, and C_3 is the straight path from $(2,1)$ back to the origin. (See Exercise 30 of Section 15.2.)
67. Find $\int_C 2xy dx + x^2 dy$ along the path given in Exercise 66. (See Exercise 31 of Section 15.2.)
68. Determine the work done by the force field $\mathbf{F}(x, y) = \langle x + y, 2xy \rangle$ if it moves a particle clockwise around the triangle with vertices $(0,0)$, $(2,0)$, and $(1,1)$.
69. Determine the work done by the force field given in Exercise 68 as it moves a particle clockwise around the square $[-1,1] \times [-1,1]$.
70. Determine the work done by the force field $\mathbf{F}(x, y) = \langle xy, x^2 - y^2 \rangle$ as it moves a particle counterclockwise around the unit circle centered at the origin.
71. Use Green's Theorem to determine the work done by the force field $\mathbf{F}(x, y) = \langle 2, 3x \rangle$ if it moves a particle counterclockwise around the ellipse parametrized as $\mathbf{r}(t) = \langle 3 + 3\cos t, 2 + 2\sin t \rangle$, $0 \leq t \leq 2\pi$. (See Exercise 36 of Section 15.2.)

72–75 Evaluate the outward flux of the vector field \mathbf{F} across the specified path C .

72. $\mathbf{F}(x, y) = \langle xy, 2x + y \rangle$, C is the boundary of the region between the graphs of $y = 1 - x^2$ and the x -axis
73. $\mathbf{F}(x, y) = \langle x^2y, x^3 + y^2 \rangle$, C is the boundary of the region bounded by the x -axis and the graphs of $y = x^3$ and $x = 2$
74. $\mathbf{F}(x, y) = \langle 4x^2y^3, -x^3y \rangle$, C is the boundary of the region bounded by the coordinate axes and the line $y = 2 - 2x$
75. $\mathbf{F}(x, y) = \langle e^x, e^x y^2 \rangle$, C is the triangle with vertices $(0,0)$, $(1,2)$, and $(5,0)$

76–84 Suppose that R is a region bounded by a positively oriented, piecewise smooth, simple closed planar curve C . Recall that the area of R is the double integral of $f(x, y) = 1$ over R ; that is, $A = \iint_R 1 \cdot dA$.

With Green's Theorem in mind, if we choose P and Q so that

$$\frac{dQ}{dx} - \frac{dP}{dy} = 1, \text{ then the area of } R \text{ can be found from a line integral:}$$

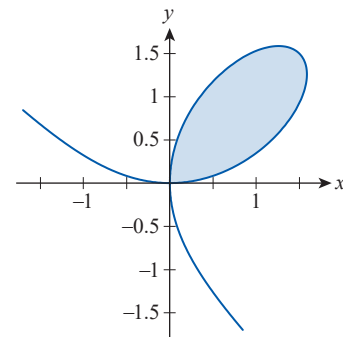
$$A = \iint_R 1 \cdot dA = \iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy} \right) dA = \oint_C P dx + Q dy.$$

In Exercises 76–84, use this observation to prove the area formula.

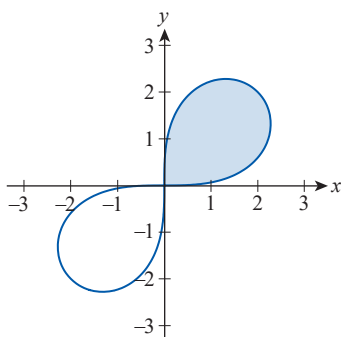
76. Show that if R and C are as above, then the area A of R is

$$A = \frac{1}{2} \oint_C x dy - y dx = \oint_C x dy = -\oint_C y dx.$$

77. Use Exercise 76 to verify the area formula of the circle $x^2 + y^2 = r^2$.
78. Use Exercise 76 to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
79. Use Exercise 76 to find the area of the region bounded by the y -axis and the lines $3y = x$ and $y = 4 - x$.
80. Use Exercise 76 to find the area of the region bounded by the graphs of $y = x^2$ and $y = x + 2$.
81. Use Exercise 76 to find the area of the region bounded by the graphs of $y = 9 - x^2$ and $3y = x^2 - 9$.
- 82.* The graph of the equation $x^3 + y^3 = 3xy$ is called the folium of Descartes. Use Exercise 76 to find the area of its loop. (**Hint:** Choose $t = y/x$ to obtain the parametrization $x = 3t/(1+t^3)$ and $y = 3t^2/(1+t^3)$. Use the Quotient Rule to obtain $d(y/x) = (x dy - y dx)/x^2$; then use Exercise 76. Note that you obtain half the area of the loop by integrating on the interval $0 \leq t \leq 1$.)



- 83.* Follow the hint provided for Exercise 82 to obtain the area of the first-quadrant loop of the lemniscate $(x^2 + y^2)^2 = 16xy$. (Note that by symmetry, doubling your answer would yield the total area enclosed by the lemniscate.)



84. Use Exercise 76 to obtain the following convenient area formula for the polygon with vertices $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$. Show that the area A of such a polygon is

$$A = \frac{1}{2} \sum_{k=1}^n (x_k y_{k+1} - x_{k+1} y_k),$$

with the convention that $(x_1, y_1) = (x_{n+1}, y_{n+1})$.

(Hint: Start by letting S be one side of the polygon, say the one connecting (x_1, y_1) with (x_2, y_2) , and show that $\frac{1}{2} \int_S x dy - y dx = \frac{1}{2} (x_1 y_2 - x_2 y_1)$. Then use Exercise 76.)

85–88 Use the formula from Exercise 84 to find the area of the given polygon.

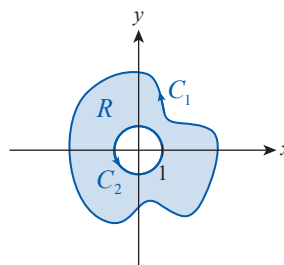
85. The square with vertices $(0,0)$, $(3,0)$, $(3,3)$, and $(0,3)$
86. The triangle with vertices $(0,0)$, $(2,4)$, and $(0,4)$
87. The hexagon with vertices $(0,0)$, $(2,0)$, $(4,2)$, $(3,4)$, $(1,4)$, and $(0,2)$
88. The (nonconvex) pentagon with vertices $(1,0)$, $(3,0)$, $(5,7)$, $(2,2)$, and $(0,3)$
89. Evaluate $\oint_C (3x^3 - y) dx + (y^2 + 4x) dy$ on the ellipse given in Exercise 78.

90. Let C_2 be the unit circle centered at the origin, let $C = C_1 \cup C_2$ be the boundary of region R with the indicated orientations as in the figure below, and let $\mathbf{F}(x, y) = \langle 4x - 3y, 3x + y^2 \rangle$. Assuming that the area of R is 4π square units, use Green's Theorem to evaluate

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}. \quad (\text{Hint: Determine } \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

and use the fact that by Green's Theorem, it equals

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}.)$$



91. In Exercise 35 of Section 15.3, you showed that for the vector field \mathbf{F} in Example 6, $\oint_{C_r} \mathbf{F} \cdot d\mathbf{r} = 2\pi$, where C_r is a circle of radius r around the origin (you were free to choose the radius in that problem, but $r = 1$ is the natural choice). Combine that result with Example 6 of this section to show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every simple closed path C enclosing the origin.

92. For the vector field given in Exercise 91, show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed path C that is disjoint from, and does not enclose the origin.

93. Use the technique illustrated in Example 6 to find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field

$$\mathbf{F}(x, y) = \frac{\langle x, y \rangle}{(x^2 + y^2)},$$

where C is a simple closed path enclosing the origin.

94. Use Green's Theorem to show that the area of a region R enclosed by the simple closed curve C in polar coordinates is

$$A = \frac{1}{2} \oint_C r^2 d\theta.$$

95. Use Green's Theorem to show that the coordinates of the centroid of a region R of constant density ρ and area A , enclosed by the simple closed curve C can be found as

$$\bar{x} = \frac{1}{2A} \oint_C x^2 dy, \quad \bar{y} = -\frac{1}{2A} \oint_C y^2 dx.$$

(Hint: Start by determining $M_x = \iint_R y \rho dA$ and $M_y = \iint_R x \rho dA$.)

- 96.* Use Green's Theorem to show that the second moments I_x and I_y of the region R given in Exercise 95 are

$$I_x = -\frac{1}{3} \oint_C \rho y^3 dx \quad \text{and} \quad I_y = \frac{1}{3} \oint_C \rho x^3 dy.$$

97. Use Exercise 95 to find the centroid of the half disk bounded by the upper semicircle $y = \sqrt{R^2 - x^2}$ and the x -axis.
98. Use Exercise 96 to find the second moments about the coordinate axes of the half disk given in Exercise 97, assuming constant density ρ .

99. Use Green's Theorem to provide a proof of the Component Test. If D is an open and simply connected region and $\mathbf{F} = \langle P, Q \rangle$ is a vector field with continuous first partials, then $\partial P / \partial y = \partial Q / \partial x$ throughout D implies \mathbf{F} is conservative.

(Hint: Show that \mathbf{F} is path independent by using Green's Theorem and demonstrating that for a simple closed path C enclosing the region R ,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0.$$

Extend this result to general closed paths.)

Concept Check

100–103 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

100. The divergence of a vector field at a point P is a vector pointing in the direction of greatest outflow from a small rectangular neighborhood.
101. The curl of a vector field is a vector field.
102. The curl of a planar (i.e., two-dimensional) vector field is undefined.
103. In applications of Green's Theorem, clockwise orientation is negative.