15.3 **Exercises**

1. Use the Component Test to verify that the vector field **F** in Example 5 is conservative.

2–5 Verify that V is a potential function for **F** and determine the line integral along the indicated curve.

2. $\mathbf{F}(x,y)=\langle y\sin y, xy\cos y + x\sin y\rangle$; $V(x,y)=xy\sin y$; $\mathbf{r}(t)=\left\langle t,\frac{\pi}{2}t\right\rangle$, $0\le t\le 1$

3. $\mathbf{F}(x,y)=\left\langle e^y\cos x+\ln y, \frac{x}{y}+e^y\sin x\right\rangle$; $V(x,y)=e^y\sin x + x\ln y$; $\mathbf{r}(t)=\left\langle \frac{\pi}{2}+\frac{\pi}{2}t, 1+(e-1)t\right\rangle$, $0\le t\le 1$

4. $\mathbf{F}(x,y,z)=\langle -yz\sin x, z\cos x, y\cos x\rangle$; $V(x,y,z)=yz\cos x$; $\mathbf{r}(t)=\left\langle \frac{\pi}{2}(t-1), \frac{\pi}{2}t, t\right\rangle$, $0\le t\le 1$

5. $\mathbf{F}(x,y,z)=\left\langle 2xy+e^{z^2}, x^2, 2xze^{z^2}\right\rangle$; $V(x,y,z)=x^2y+xe^{z^2}$; $\mathbf{r}(t)=\langle t,t^2,t^3\rangle$, $0\le t\le 1$

6–21 Either find a potential function for **F** or state that a potential function does not exist. The latter implies that **F** is not conservative.

6. $\mathbf{F}(x,y)=\langle 2x, 2+6y\rangle$

7. $\mathbf{F}(x,y)=\langle 3x^2+2y^2, 4xy+3y^2\rangle$

8. $\mathbf{F}(x,y)=\langle x^2y, 3x\rangle$

9. $\mathbf{F}(x,y)=\langle x^2+y, 2xy+3y^2\rangle$

10. $\mathbf{F}(x,y)=\left\langle x^2+\ln y, \frac{x}{y}+2y^3\right\rangle$

11. $\mathbf{F}(x,y)=\left\langle e^x+y\cos x, \frac{1}{y}+\sin x\right\rangle$

12. $\mathbf{F}(x,y)=\langle e^x+\sin y, e^y+\cos x\rangle$

13. $\mathbf{F}(x,y)=\langle 4x^3y-y^5, x^4-5xy^4\rangle$

14. $\mathbf{F}(x,y)=\left\langle -\frac{2y}{x^3}, \frac{1}{x^2}+\frac{1}{\sqrt{y}}\right\rangle$

15. $\mathbf{F}(x,y,z)=\langle 2xyz, x^2z, x^2y\rangle$

16. $\mathbf{F}(x,y,z)=\langle yz, xz-z\sin y, xy+\cos y\rangle$

17. $\mathbf{F}(x,y,z)=\langle x^2y, 2xz, z^3\rangle$

18. $\mathbf{F}(x,y,z)=\left\langle \frac{zy^2}{x}, 2zy\ln x, -\frac{y}{x^2}\right\rangle$

19. $\mathbf{F}(x,y,z)=\langle \tan z, 2yz, y^2+x\sec^2 z\rangle$

20. $\mathbf{F}(x,y,z)=\langle y^2+ze^{xz}, 2xy-2z, xe^{xz}-2y\rangle$

21. $\mathbf{F}(x,y,z)=\left\langle x\cos z+z, z+\sin y, -\frac{x^2}{2}\sin z+x\right\rangle$

22. For the vector field **F** in Example 3, show that $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=2$ and $\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\pi-\frac{2}{3}$, meaning that **F** is not conservative.

23–26 Show that the line integral is not path independent by finding two different values for the integral along two different paths connecting A and B. (Answers may vary.)

23. $\int_C y\,dx+4\,dy$; $A(1,0)$, $B(0,1)$

24. $\int_C xy\,dx+2y\,dy$; $A(0,0)$, $B(2,4)$

25. $\int_C y\,dx-2z\,dy+2\,dz$; $A(-5,0,0)$, $B(3,4,0)$

26. $\int_C xy\,dx+z\,dy+(x+z)\,dz$; $A(0,0,0)$, $B(1,1,1)$

27–32 Show that the force field **F** is conservative, and use this fact to determine the work done by **F** in moving an object from A to B.

27. $\mathbf{F}(x,y)=\langle 3y, 3x-2y\rangle$; $A(-1,0)$, $B(5,3)$

28. $\mathbf{F}(x,y)=\langle x-y^2, -2xy\rangle$; $A(1,4)$, $B(3,-2)$

29. $\mathbf{F}(x,y)=\langle e^y-2xy, x(e^y-x)\rangle$; $A(-4,1)$, $B(0,0)$

30. $\mathbf{F}(\mathbf{r})=\frac{k\mathbf{r}}{r^3}$; $A(-2,1,-2)$, $B(6,0,-8)$

31. $\mathbf{F}(x,y,z)=\langle e^y\cos x-yz, e^y\sin x-xz, -xy\rangle$; $A(\pi,1,2/\pi)$, $B(\pi/2,0,0)$

32. $\mathbf{F}(x,y,z)=\langle \tan y, x\sec^2 y-z, -y\rangle$; $A(0,0,4/\pi)$, $(2,\pi/4,0)$

33. Find an "easier" solution (one that uses the Fundamental Theorem for Line Integrals) for Exercise 29 of Section 15.2.

34. Repeat Exercise 33, this time for Exercise 44 of Section 15.2.

35. **a.** Show that the vector field

$$\mathbf{F}(x,y)=\left\langle -\frac{y}{|\langle x,y\rangle|^2},\frac{x}{|\langle x,y\rangle|^2}\right\rangle$$

satisfies the Component Test, but is not conservative. Does this contradict the Component Test? (Such a vector field is sometimes called a *rotation field*. **Hint:** To show that **F** is not conservative, calculate a line integral along a circle centered at the origin.)

b. Verify that $\mathbf{F}(x,y)=\nabla\left(-\arctan\frac{x}{y}\right)$. Reconcile this observation with part a.

36. In this exercise, we will consider the vector field

$$\mathbf{F}(x,y)=\left\langle \frac{x}{|\langle x,y\rangle|^2},\frac{y}{|\langle x,y\rangle|^2}\right\rangle,$$

with the usual notation $P(x,y)=x/|\langle x,y\rangle|^2$ and $Q(x,y)=y/|\langle x,y\rangle|^2$.

a. Show that $\partial P/\partial y=\partial Q/\partial x$ throughout the domain of **F**.

b. Explain why **F** does not satisfy the conditions of the Component Test.

c. Show that **F** is conservative. (See Exercise 58 of Section 15.1.) Is this a contradiction with part b.?

37–42 Decide whether the specified region D is simply connected.

37. $D=\{(x,y)\,|\,x^2+y^2<1\}$

38. $D=\{(x,y)\,|\,0<x^2+y^2<1\}$

39. $D=\{(x,y)\,|\,|x|+|y|<1\}$

40. $D=\{(x,y)\,|\,x^2\le y\le 2x^2\}$

41. $D=\{(x,y,z)\,|\,1<x^2+y^2+z^2<3\}$

42. $D=\{(x,y,z)\,|\,1<z^2<3\}$

43–46 In physics, the Law of Conservation of Energy states that if an object moves under the influence of a conservative force field, then the sum of its potential and kinetic energies (the energies resulting from the object's position and motion, respectively) remains constant. For example, in case of an object falling under the influence of gravity only, the kinetic energy it gains as a result of increasing speed equals the loss in potential energy stemming from loss of altitude.

In Exercises 43–46, you will use the Fundamental Theorem for Line Integrals to derive this law.

43. Suppose an object of mass m is under the influence of a conservative force field **F**. If $\mathbf{F}=\nabla f$, the potential energy of the object is defined as

$$E_p(x,y,z)=-f(x,y,z).$$

Show that the work W done by **F** in moving the object from point A to point B along a smooth curve is

$$W=E_p(A)-E_p(B).$$

44. Referring to Exercise 43, suppose the path of the object is parametrized by $\mathbf{r}(t)$ so that $\mathbf{r}(a)=A$ and $\mathbf{r}(b)=B$. Show that W can be written as

$$W=\int_a^b \mathbf{F}\cdot\mathbf{v}(t)\,dt,$$

where $\mathbf{v}(t)$ is the velocity of the object.

45. Use Newton's Second Law $(\mathbf{F}=m\mathbf{v}'(t))$ along with Exercise 44 to show that W can be expressed as

$$W=\frac{m}{2}\int_a^b \frac{d}{dt}\left(|\mathbf{v}(t)|^2\right)dt.$$

46. Use Exercise 45 and the fact that the kinetic energy of an object of mass m and speed $v=|\mathbf{v}|$ is $E_k=\frac{1}{2}mv^2$ to conclude that

$$W=E_k(B)-E_k(A).$$

Consequently, using Exercise 43,

$$E_p(A)+E_k(A)=E_p(B)+E_k(B).$$

47. Use the Law of Conservation of Energy to derive the formula for the velocity of impact of an object falling from height h under the influence of gravity only, $v_{imp}=\sqrt{2hg}$. (**Hint:** Since we are ignoring all other forces, the initial potential energy $E_p=mgh$ turns entirely into kinetic energy.)

48.* A proton is moving along the z-axis in the positive direction at a speed of $2 \cdot 10^6$ m/s (assume units are meters in our coordinate system). At the origin, it encounters an electric force field $\mathbf{E}(x, y, z) = \langle 0, 0, 1600z \rangle$ N/C (newtons per coulomb). Use the Law of Conservation of Energy to find the proton's speed at the point $(0, 0, 6)$. (**Hint:** For the mass and charge of a proton, use the approximate data $m = 1.6726 \times 10^{-27}$ kg and $q = 1.6 \times 10^{-19}$ C, respectively. The potential energy of the proton at (x, y, z) is $E_p(x, y, z) = qV(x, y, z)$, where V is the electric potential.)

49. Suppose $f(x, y)$ is a harmonic function. Prove

$$\int_C \left(\frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy \right) = 0$$

along any smooth closed curve C in $\mathbb{R}^2$. (For a refresher on the definition of harmonic functions, see Exercise 98 of Section 13.3.)

Concept Check

50–54 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

50. If $\mathbf{F}$ is path independent on an open connected region D, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every path C in D.

51. If the components of $\mathbf{F} = \langle P, Q \rangle$ have continuous first partials and $\partial P/\partial y = \partial Q/\partial x$ throughout an open connected region D, then $\mathbf{F} = \langle P, Q \rangle$ is conservative on D.

52. If $\mathbf{F}$ is continuous on an open connected region D and every line integral $\int \mathbf{F} \cdot d\mathbf{r}$ is path independent, then $\mathbf{F}$ is conservative.

53. If $\mathbf{F}$ is conservative on an open connected region D, then every line integral $\int \mathbf{F} \cdot d\mathbf{r}$ is path independent in D.

54. The domain of the vector field $\mathbf{F}(\mathbf{r}) = \frac{k\mathbf{r}}{r^3}$ is not simply connected; therefore, it cannot be conservative.