15.2 Exercises

1. Determine $\int_C xy^2\,ds$, where C is the first-quadrant portion of the circle $x^2+y^2=4$ traversed counterclockwise. (See Example 1 for a hint on parametrization.)

2. Show that you obtain the same answer as that in Exercise 1 if you parametrize the quarter circle as $x=t,\ y=\sqrt{4-t^2},\ 0\le t\le 2$.

3–7 Evaluate the indicated line integral.

3. $\int_C (xy+1)\,ds$, where C is the lower semicircle $y=-\sqrt{9-x^2}$, traversed counterclockwise

4. $\int_C y\,ds$, where C is the graph of $y=\sqrt{x}$, traversed from $(0,0)$ to $(1,1)$

5. $\int_C (2x+y)\,ds$, where C is the line segment from the origin to the point $(1,\sqrt{3})$, followed by the arc of the circle $x^2+y^2=4$ traversed counterclockwise from $(1,\sqrt{3})$ to $(-2,0)$

6. $\int_C (x+3y)\,ds$, where C is the line segment from $(0,1)$ to $(-2,2)$, followed by the arc of the circle $x^2+y^2=8$ traversed clockwise from $(-2,2)$ to $(2\sqrt{2},0)$

7. $\int_C \sqrt{1+18xy}\,ds$, where C is the graph of $y=2x^3$, $0\le x\le 1$

8. If a piece of wire is bent into the semicircle $y=\sqrt{16-x^2}$ and its density function is $\rho(x,y)=2x^2+y^2$, find the mass and center of mass of the wire. (See Example 1 for a hint on parametrization.)

9. Repeat Exercise 8 for a wire that is bent into the upper semicircle of radius R, $y=\sqrt{R^2-x^2}$, so that the density at any point is proportional to the distance from the line $y=R$.

10. Find the mass of the wire in Exercise 8 by using the parametrization $x=4\cos(t^2),\ y=4\sin(t^2)$, $0\le t\le\sqrt{\pi}$. Verify that you obtain the same answer as in Exercise 8.

11. Find the mass of the wire bent into a parabolic arc $y=\sqrt{x}, 1\le x\le 3$, if its density is $\rho(x,y)=2x/y$.

12–17 Evaluate the indicated line integral.

12. $\int_C (x+2y+z^2)\,ds$, where C is the line segment joining the origin and the point $(1,2,3)$

13. Integrate the function given in Exercise 12 along the path that is the line segment joining the origin with the point $(1,2,0)$, followed by the segment from $(1,2,0)$ to $(1,2,3)$. Is your answer equal to that given for Exercise 12?

14. $\int_C (2x+yz)\,ds$, where C is the line segment joining $(1,2,0)$ and $(3,4,1)$

15. $\int_C (y+16)\,ds$, where C can be parametrized as $\mathbf{r}(t)=\left\langle \frac{1}{2}t^2, 2t, \frac{8}{3}t^{3/2}\right\rangle,\ 0\le t\le 2$

16. $\int_C (3x^2y+z)\,ds$, where C is the helix $\mathbf{r}(t)=\langle \cos t, \sin t, t/2\rangle$, for $0\le t\le 8\pi$

17. $\int_C ye^z\,ds$, where C is the helix given in Exercise 16

18. Use a line integral to find the length of the helix given in Exercise 16.

19. Find the mass of the helix given in Exercise 16 if its density is proportional to the distance from the xy-plane.

20. Find the center of mass of the helix given in Exercise 19.

21. Evaluate $\int_C x\,ds$ on the curve C parametrized by $\mathbf{r}(t)=\langle \cos t, \sin t, \cos t\rangle,\ \ 0\le t\le \pi/2$.

22. Determine the moments of inertia and radii of gyration for the object in Example 3.

23. Determine the mass and the center of mass of the V-shaped object in Example 3 if its density is proportional to the distance from the xy-plane.

24. Determine the moments of inertia and radii of gyration for the object in Exercise 23.

25. Find the mass of the spring that is defined by $\mathbf{r}(t)=\langle t, 3\cos t, 3\sin t\rangle,\ \ 0\le t\le 4\pi$, if its density function is $\rho(x,y,z)=x/2$.

26–30 Evaluate the line integral of the vector field along the given curve.

26. $\int_C x^2 y\,dx + (y^2 - x^2)\,dy$, where C can be parametrized as $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$

27. $\int_C (x^2 y + xy^2)\,dx + x^3\,dy$, where C is the unit circle centered at the origin

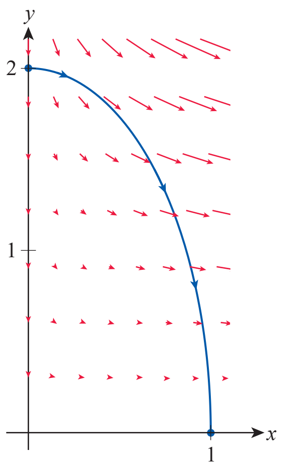
28. $\int_C y\,dx + xy\,dy$, where **a.** $C = C_1$: the line segment joining the origin with $(1,1)$, **b.** $C = C_2$: the parabola $y = x^2$ joining the origin with $(1,1)$, and **c.** $C = C_3$: the parabola $x = y^2$ joining the origin with $(1,1)$. Do your answers differ?

29. Repeat Exercise 28 for the line integral $\int_C 2xy\,dx + x^2\,dy$. Compare your answers.

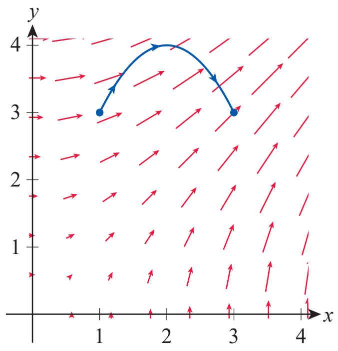
30. $\int_C (x^2 + y)\,dx + xy\,dy$ along the closed path $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from the origin to $(1,0)$, C_2 is the line segment from $(1,0)$ to $(2,1)$, and C_3 is the straight path from $(2,1)$ back to the origin

31. Find $\int_C 2xy\,dx + x^2\,dy$ along the path given in Exercise 30.

32. Determine the work done by the force field in Example 4 if it moves a particle along the elliptical path $y = \sqrt{4 - 4x^2}$ from $(0,2)$ to $(1,0)$.

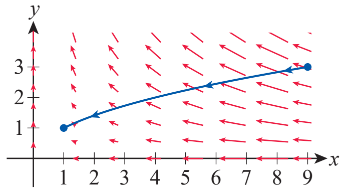


33. Determine the work done by the force field $\mathbf{F}(x,y) = \langle y, x \rangle$ if it moves a particle along the parabolic arc $y = 4x - x^2$ from the point $(1,3)$ to $(3,3)$.



34. By parametrizing the path given in Exercise 33 oriented from $(3,3)$ toward $(1,3)$, find the work done on the particle by the force field when it travels along this "reverse path." Explain your findings.

35. Determine the work done by the force field $\mathbf{F}(x,y) = \langle -x, y \rangle$ if it moves a particle along the graph of $y = \sqrt{x}$ from the point $(9,3)$ to $(1,1)$.



36. Determine the work done by the force field $\mathbf{F}(x,y) = \langle 2, 3x \rangle$ if it moves a particle counterclockwise around the ellipse parametrized as $\mathbf{r}(t) = \langle 3 + 3\cos t, 2 + 2\sin t \rangle$, $0 \le t \le 2\pi$.

37–40 Evaluate the indicated line integral.

37. $\int_C xy\,dx + z\,dy + (x + z)\,dz$, where C can be parametrized as $\mathbf{r}(t) = \langle t^2, t, t^3 \rangle$, $0 \le t \le 1$

38. The integral of Exercise 37 along the line segment from the origin to $(2,2,0)$, followed by the segment from $(2,2,0)$ to $(4,6,2)$

39. $\int_C (x - y^2)\,dx + (y - z^2)\,dy + (z - x^2)\,dz$, where C is the "twisted cube" $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$

40. $\int_C y\,dx + z\,dy + x^2\,dz$, where C is parametrized by $\mathbf{r}(t) = \langle t^2, t, e^t \rangle$, $0 \le t \le 1$

41. Determine the work done by the force field $\mathbf{F}(x,y,z) = \langle x, y, 1 \rangle$ on a particle that moves up the helix given in Exercise 16.

42. Determine the work done by the force field $\mathbf{F}(x,y,z)=\langle x,z,y\rangle$ on a particle that moves along the curve $\mathbf{r}(t)=\langle t^2,\cos t,\sin t\rangle$, $0\le t\le\pi$.

43. Find the work done by the force field $\mathbf{F}=-\frac{k\mathbf{r}}{r^3}$ as it moves a particle from the point $(0,0,4)$ to $(0,3,4)$ along a straight-line curve. (For well-known force fields of this type, see Examples 3 and 4 of Section 15.1.)

44. Determine the work done by the force field $\mathbf{F}(x,y,z)=\langle y,x+z,y\rangle$ on a particle that moves around the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in a counterclockwise direction (when viewed from the point $(1,1,1)$).

45–48 A fluid's velocity field is given by $\mathbf{F}(x,y,z)$. Determine the fluid's flow along the indicated curve $\mathbf{r}(t)$.

45. $\mathbf{F}(x,y,z)=\langle 1,2xz,4y\rangle$; $\mathbf{r}(t)=\langle 1,t,t^2\rangle$; $0\le t\le 3$

46. $\mathbf{F}(x,y,z)=\langle xy,yz,zx\rangle$; $\mathbf{r}(t)=\langle 1,2t,3t\rangle$; $0\le t\le 2$

47. $\mathbf{F}(x,y,z)=\langle -2y,y-x,3\rangle$;
$\mathbf{r}(t)=\langle 4\cos t,2\sin t,t\rangle$; $0\le t\le\pi$

48. $\mathbf{F}$ is the gradient of the potential function $f(x,y,z)=x^2+y^2+z^2$, and $\mathbf{r}$ is the helix given in Exercise 16.

49. **a.** Verify that $f(x,y,z)=\frac{1}{2}x^2y^2z$ is a potential function for the following vector field.
$\mathbf{F}(x,y,z)=\left\langle xy^2z,x^2yz,\frac{1}{2}x^2y^2\right\rangle$

b. Assuming that $\mathbf{F}$ is the velocity field of a fluid, find the circulation of the fluid around the unit circle $\mathbf{r}(t)=\langle\cos t,0,\sin t\rangle$, $0\le t\le 2\pi$.

50–53 In physics, the electrostatic potential at a point P resulting from a single point charge q is calculated using the formula

$$V(P)=\frac{\varepsilon q}{r_P},$$

where r_P is the distance between P and the point charge q, and ε is Coulomb's constant (see Example 4 of Section 15.1). The value of ε is approximately $8.988\times10^9\ \mathrm{Nm^2/C^2}$. We note that Coulomb's constant is often used in the form $\varepsilon=1/(4\pi\varepsilon_0)$, where $\varepsilon_0\approx 8.85\times10^{-12}\ \mathrm{C^2/Nm^2}$. The constant ε_0 is called the *permittivity constant*. (We assume here that the potential at an "infinitely distant" point is zero.)

Among their many uses, line integrals enable us to calculate the electrostatic potential around a continuously charged curve, according to the following formula.

Suppose a curve C has continuous charge distribution given by its charge density function $q(x,y,z)$. The electrostatic potential at a point P is then obtained from

$$V(P)=\varepsilon\int_C\frac{q(x,y,z)}{r_P(x,y,z)}ds,$$

where $r_P(x,y,z)$ is the distance between (x,y,z) and P. We will use the above formula in Exercises 50–53.

50. Suppose the quarter circle in Example 4 has charge density $q(x,y,z)=\frac{1-y}{10^7}$ coulomb per meter (C/m). Find the electrostatic potential at the point $(0,0,1)$.

51. Repeat Exercise 50 if the charge density is $q(x,y,z)=\frac{xy}{10^5}$ C/m.

52. Suppose the segment of the x-axis between the origin and the point $(2,0,0)$ has charge density $q(x,y,z)=\frac{x}{10^4}$ C/m. Find the electrostatic potential at the point $(0,0,1)$.

53.* Find the potential at the origin of the electric field created by the helix $\mathbf{r}(t)=\langle\cos t,\sin t,t\rangle$, $0\le t\le 2\pi$, with charge density $q(x,y,z)=\frac{1-z}{10^6}$ C/m.

15.2 Technology Exercises

54–59 Use technology to help with your calculations. Express your answers as decimal approximations.

54. Find the mass and the centroid of the thin wire given by $y=9-x^2$, $-3\le x\le 3$, if it has constant density ρ.

55. Determine the three moments of inertia and radii of gyration for the wire in Exercise 54.

56. Repeat Exercise 54 if the wire has density $\rho(x,y)=x^2y$.

57. Repeat Exercise 55 if the wire has density $\rho(x,y)=x^2y$.

58. Determine the three moments of inertia and radii of gyration for the spring given in Exercise 25.

59. Find the electrostatic potential at $(1,0,0)$ if the wire in Exercise 54 has charge density $q(x,y,z)=\frac{y}{10^6}$ C/m.