15.1 Exercises

1–6 Identify the real-life function as a vector field or a scalar field.

1. Fluid pressure in a swimming pool

2. Velocity of the wind inside a hurricane

3. Electromagnetic force around a coil

4. Air temperature near a working hair dryer

5. Velocity of water flowing through a pipe

6. CO concentration around a barbecue grill

7–10 Evaluate the vector field at the given point.

7. $\mathbf{F}(x,y)=\langle x+y,-2x\rangle$; $(1,-1)$

8. $\mathbf{F}(x,y)=\langle xe^x, xy^2+1\rangle$; $(2,1)$

9. $\mathbf{F}(x,y,z)=\left\langle z,1,\sqrt{x^2+y^2+z^2}\right\rangle$; $(3,4,0)$

10. $\mathbf{F}(x,y,z)=\left\langle x-z,\dfrac{xy}{z^2},yz\right\rangle$; $(1,1,2)$

11–14 Match the given two-dimensional (planar) vector field with its graph (labeled A–D).

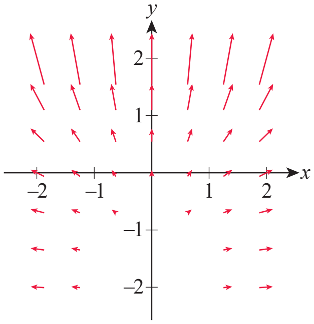
11. $\mathbf{F}(x,y)=\langle y,-x\rangle$

12. $\mathbf{F}(x,y)=\langle x,-y\rangle$

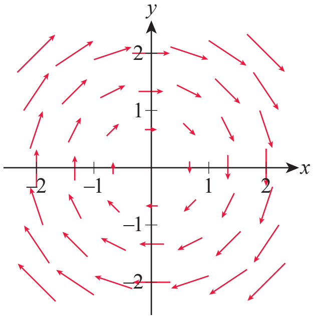
13. $\mathbf{F}(x,y)=\langle x,e^y\rangle$

14. $\mathbf{F}(x,y)=\dfrac{\langle x-1,y\rangle}{|\langle x-1,y\rangle|}$

A.



B.



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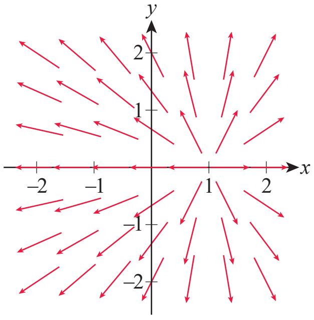
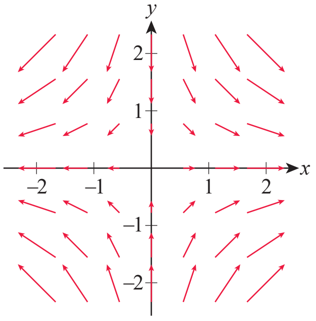


D.



15–23 Sketch the vector field **F** by hand, using a sufficient number of representative vectors.

15. $\mathbf{F}(x,y)=\langle 1,0\rangle$

16. $\mathbf{F}(x,y)=\langle 0,y\rangle$

17. $\mathbf{F}(x,y)=\langle -4x,0\rangle$

18. $\mathbf{F}(x,y)=\dfrac{\langle -y,x\rangle}{|\langle -y,x\rangle|}$

19. $\mathbf{F}(x,y)=\dfrac{\langle x,-y\rangle}{|\langle x,-y\rangle|}$

20. $\mathbf{F}(x,y)=\langle 1,y-1\rangle$

21. $\mathbf{F}(x,y)=\langle y+2,1-x\rangle$

22. $\mathbf{F}(x,y)=\langle x,y^2\rangle$

23. $\mathbf{F}(x,y)=\dfrac{\langle x,y\rangle}{|\langle x,y\rangle|}$

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24–27 Match the given three-dimensional vector field with its graph (labeled A–D).

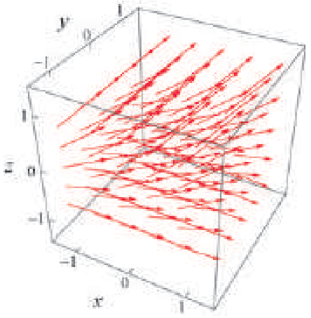
24. $\mathbf{F}(x,y,z)=\langle 1,0,1\rangle$

25. $\mathbf{F}(x,y,z)=\langle 1,1,z\rangle$

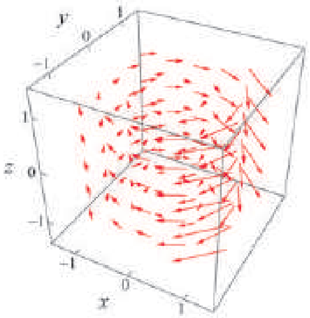
26. $\mathbf{F}(x,y,z)=\langle x,y,1\rangle$

27. $\mathbf{F}(x,y,z)=\dfrac{\langle y,-x,0\rangle}{2-x}$

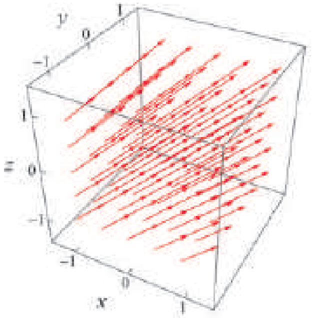
A.



B.



C.



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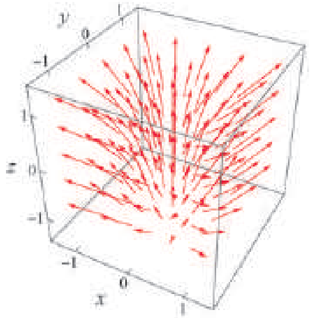

28–36 Determine ∇f for the given scalar field f.

28. $f(x,y)=\dfrac{x^2y}{2}+xy^3$

29. $f(x,y)=\ln\sqrt{x^2+y^2}$

30. $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$

31. $f(x,y)=y^2\tan^{-1}x$

32. $f(x,y)=ye^{x+y}$

33. $f(x,y,z)=ze^{xyz}$

34. $f(x,y,z)=y^2z-2x^2z^2$

35. $f(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$

36. $f(x,y,z)=\dfrac{xy}{y-z}$

37. By determining a potential function f, show that the electric field $\mathbf{F}(x,y,z)=\dfrac{\varepsilon Qq}{r^3}\mathbf{r}$ is conservative (i.e., $\mathbf{F}=\nabla f$ for some potential function f).

38. Prove that if the vector field $\mathbf{F}=\langle P,Q,R\rangle$ is conservative (i.e., $\mathbf{F}=\nabla f$ for some potential function f), then $\mathbf{F}$ satisfies what is sometimes called the *Component Test*, that is,

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}.$$

(As we will see in Section 15.7, if the domain of $\mathbf{F}$ satisfies certain regularity conditions, then the converse of the above statement also holds. **Hint:** Start by noting that $P=\partial f/\partial x$ and $Q=\partial f/\partial y$. Use this to write both $\partial P/\partial y$ and $\partial Q/\partial x$ as mixed partials of f, and use Clairaut's Theorem (see Section 13.3) to obtain the first statement. Use a similar approach to prove the last two equalities.)

39. Formulate and prove a result analogous to the one in Exercise 38 for the vector field $\mathbf{F}=\langle P,Q\rangle$.

40–47 Use Exercises 38 and 39 to decide whether the given vector field is conservative.

40. $\mathbf{F}(x,y)=\langle y^2,2xy\rangle$ **41.** $\mathbf{F}(x,y)=\langle y,xy\rangle$

42. $\mathbf{F}(x,y)=\langle xe^y,ye^x\rangle$

43. $\mathbf{F}(x,y)=\langle y^2\cos(xy^2),2xy\cos(xy^2)\rangle$

44. $\mathbf{F}(x,y,z)=\langle 1,xy,0\rangle$

45. $\mathbf{F}(x,y,z)=\langle y^2z,2xyz,xy^2\rangle$

46. $\mathbf{F}(x,y,z)=\dfrac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}$

47. $\mathbf{F}(x,y,z)=\left\langle \dfrac{z}{\sqrt{xy}},-\dfrac{x}{\sqrt{xy}},\sqrt{xyz}\right\rangle$

48–53 Consider $\mathbf{F}(x,y)=\langle 12xy,6x^2+2\rangle$, a conservative vector field. We can find a potential function f by integrating, as follows. First, note that we must have $\partial f/\partial x=12xy$, so we integrate with respect to x to obtain $f(x,y)=6x^2y+g(y)$, where g is a function of y (or a constant). On the other hand, we know $\partial f/\partial y=6x^2+2$, so differentiating with respect to y the function f we obtained above, we see that $6x^2+2=\partial f/\partial y=6x^2+g'(y)$, that is, it must be the case that $g'(y)=2$. This implies $g(y)=2y+C$, hence $f(x,y)=6x^2y+2y+C$ is a potential function for $\mathbf{F}$ under any choice of the constant C.

Use the above technique to find a potential function for the given conservative vector field. (In Exercises 52–53, generalize to three variables.)

48. $\mathbf{F}(x,y)=\langle 4y^3,12xy^2\rangle$

49. $\mathbf{F}(x,y)=\langle y^2e^{xy^2},2xye^{xy^2}\rangle$

50. $\mathbf{F}(x,y)=\langle 3x^2-2y^2,2y-4xy\rangle$

51. $\mathbf{F}(x,y)=\left\langle \dfrac{x}{\sqrt{x^2+y^2}},\dfrac{y}{\sqrt{x^2+y^2}}\right\rangle$

52. $\mathbf{F}(x,y,z)=\langle y^2,2xy,2z\rangle$

53. $\mathbf{F}(x,y,z)=\langle 2x+z,2z-1,x+2y\rangle$

54. Using the notation in Example 3, let $\mathbf{r}=\langle x,y\rangle$ and $r=|\langle x,y\rangle|$, and describe the vector field

$$\mathbf{e}_r=\frac{\langle x,y\rangle}{|\langle x,y\rangle|}=\frac{\mathbf{r}}{r}$$

as well as its three-dimensional analogue (also denoted $\mathbf{e}_r$). See Example 1 for guidance.

55. Show that $\nabla r=\mathbf{e}_r$ holds both in two and three dimensions. (Note that you have already provided a constructive existence proof for the potential function of $\mathbf{e}_r$ in the two-dimensional case in Exercise 54.)

56. Find formulas for ∇r^2 and ∇r^3.

57. Find and prove a formula for ∇r^n $(n\in\mathbb{N})$.

58. Show that the vector field $\mathbf{r}/r^2$ is conservative by determining a potential function. (Handle both the planar and three-dimensional cases.)

59. Repeat Exercise 58 to find a potential function for the vector field $\mathbf{r}/r^3$. (Compare your finding with the results in Example 6 and Exercise 37.)

60.* Generalize Exercise 59 by determining a formula for the potential function of the vector field $\mathbf{r}/r^n$, $n\in\mathbb{N}$. Prove your assertion.

61–62 The *flow lines* of a vector field $\mathbf{F}$ are "paths aligned with $\mathbf{F}$," more precisely, paths whose *velocity field* is $\mathbf{F}$ when followed by a particle. Even more precisely, a flow line is a path $\mathbf{r}(t)$ such that $\mathbf{r}'(t)=\mathbf{F}(\mathbf{r}(t))$. In these exercises, you will determine the flow lines of vector fields.

61. By visualizing its graph, try to predict what the flow lines of the vector field $\mathbf{F}(x,y)=\langle y,-x\rangle$ might look like (see Exercise 11). Then use a differential equation to determine the equations of these flow lines. (**Hint:** Note that if $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ is a flow line of $\mathbf{F}$, then $dx/dt=y$ and $dy/dt=-x$. Conclude that $dy/dx=-x/y$ and solve the differential equation. For a refresher on separable differential equations, see Section 8.1.)

62. Repeat Exercise 61 for the vector field of Exercise 12.

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