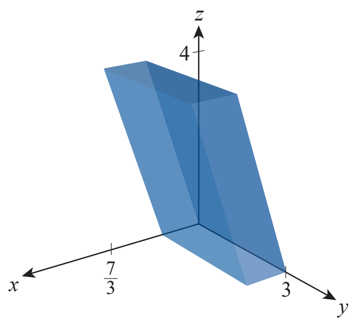

Figure 6

Solution

As in Example 2, the first step is to decide on a change of variables to try. On the basis of the integrand, and also guided by the form of two of the six planes, the assignment $u = 3x - z$ appears promising. This will allow two of the faces of the parallelepiped to be expressed as $u = 0$ and $u = 3$. In Exercise 41, you will show that if we let $v = z/2$ and $w = y/3$, then

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = 2$$

and

$$\iiint_R \left(\frac{3x-z}{2} + \frac{y}{3}\right) dV = \int_{w=0}^{w=1}\int_{v=0}^{v=2}\int_{u=0}^{u=3}\left(\frac{u}{2}+w\right)(2)\,du\,dv\,dw = 15.$$

The solid S corresponding to the limits, in xyz-space, appears in Figure 6.

14.6 Exercises

1–4 Evaluate the given determinant.

1. $\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$ **2.** $\begin{vmatrix} 1 & 6 \\ -4 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & 0 \\ 1 & -5 & 2 \end{vmatrix}$ **4.** $\begin{vmatrix} 2 & -2 & -1 \\ 3 & 2 & 3 \\ -4 & 1 & -1 \end{vmatrix}$

5–13 Find the Jacobian of the given transformation.

5. $x = 2u + v, \quad y = v - u$ **6.** $x = v, \quad y = 4u + \dfrac{v}{2}$

7. $x = 4uv, \quad y = u + 2v$ **8.** $x = u + 2v^2, \quad y = uv$

9. $x = u^2, \quad y = \dfrac{v}{u}$ **10.** $x = e^{-u}, \quad y = ve^u$

11. $x = uv, \quad y = (u+1)v$

12. $x = e^v \cos u, \quad y = e^v \sin u$

13. $x = u\cos\varphi + v\sin\varphi, \quad y = u\sin\varphi - v\cos\varphi$

14. A transformation $T(u,v) = (au + cv, bu + dv)$ is called **linear**, where a, b, c, and d are constants. (See Exercises 5 and 6.) Find a general formula for the Jacobian of a linear transformation.

15–19 Consider the parallelogram P in the xy-plane bounded by the lines $y = x + 2$, $y = x - 4$, $y = 1 - 3x$, and $y = 5 - 3x$. We can identify a linear transformation $T(u,v)$ mapping a rectangle in the uv-plane onto P as follows. Rewrite the equations of the lines as $y - x = 2$, $y - x = -4$, $y + 3x = 1$, and $y + 3x = 5$, respectively. Then perform the change of variables $u = y - x$, $v = y + 3x$. Solve for x and y to obtain T (Exercise 15), and note that the preimage of P under T is the rectangle $-4 \le u \le 2$, $1 \le v \le 5$. In Exercises 16–19, you will be asked to follow these steps to identify the indicated linear transformation.

15. By solving the above system for x and y, find $T(u,v)$, as suggested by the above directions.

Find a linear transformation $T(u,v)$ that maps a rectangular region onto the given parallelogram P.

16. P is bounded by $2y = 1 - x$, $2y = 3 - x$, $y = 3x$, and $y = 3x + 4$.

17. P is bounded by $y = \frac{3}{2}x + 2$, $y = \frac{3}{2}x + 4$, $y = 1 - \frac{1}{4}x$, and $y = 4 - \frac{1}{4}x$.

18. P is bounded by $y = 1 - 2x$, $y = 5 - 2x$, $y = 3x - 2$, and $y = 3x + 3$.

19. P is bounded by $y = 2x$, $y = 2x + 4$, $y = -2x$, and $y = -2x - 2$.

20–25 Use a change of variables in order to integrate on an appropriate rectangle. (See Exercises 16–19.)

20. Find the area of the parallelogram in Exercise 16. (**Hint:** Start with $A = \iint_P dA$, and change variables, so you can integrate on the rectangle $[1,3]\times[0,4]$.)

21. Evaluate $\iint_P (2y+x)\,dA$ on P of Exercise 17.

22. Evaluate $\iint_P (x-2y)^2\,dA$ on P of Exercise 18.

23. Evaluate $\iint_P \frac{y-3x}{4y+2x}\,dA$ on P of Exercise 16.

24. Evaluate $\iint_P \cos(x+y)\,dA$ on P of Exercise 18.

25. Evaluate $\iint_P \frac{e^{y-3x}}{x+2y}\,dA$ on P of Exercise 16. (Note that a change of variables is necessary here, since you can't find an antiderivative in Cartesian coordinates, no matter what the order of integration is.)

26. Use a change of variables to evaluate $\iint_R \frac{x}{y+2x}\,dA$, where R is the region bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$.

27. Let R be the region bounded by the coordinate axes and the line $x + 2y = 2$. Use the method of Example 2 to evaluate $\iint_R \frac{x+2y}{x+y}\,dA$.

28. Use a change of variables to evaluate $\iint_R \frac{2x+y}{8x+6y}\,dA$, where R is the region bounded by the coordinate axes and the line $y + 2x = 2$.

29. Consider the transformation $x = au$, $y = bv$ $(a, b > 0)$. Show that it maps the interior of the circle $u^2 + v^2 = 1$ onto that of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Determine the Jacobian and use a change of variables to prove the area formula for the ellipse, $A = \pi ab$.

30–33 Let T be the coordinate transformation $x = uv$, $y = v/u$, assuming $u, v > 0$.

30. Find the Jacobian for the above transformation T.

31. Show that the T-images of vertical lines $u = a$ are lines through the origin, while horizontal lines $v = b$ are mapped onto branches of hyperbolas. (Use the observations $xy = v^2$ and $x/y = u^2$.)

32. Find the T-image of the uv-rectangle $[1,\sqrt{2}]\times[1,\sqrt{2}]$. (See Exercise 31.)

33. If S denotes the T-image of the rectangle in Exercise 32, use a change of variables to evaluate $\iint_R 2x^2 y\,dy\,dx$.

34. Use a change of variables to evaluate $\iint_R xy^3\,dy\,dx$, where R is the region in the xy-plane bounded by the horizontal lines $y = 1$, $y = 3$, and the hyperbolas $y = 1/x$ and $y = 6/x$. (**Hint:** Consider the coordinate transformation $x = u/v$, $y = v$.)

35. Solve Exercise 34 without changing variables and compare your answers.

36. Generalize Exercise 14 to find a formula for the Jacobian J of a linear transformation in the three-variable case.

37–38 Find the Jacobian of the indicated coordinate transformation.

37. $x = u + 2v, \quad y = 2u + v - w, \quad z = 2v + w$

38. $x = u^2 v, \quad y = 2uvw, \quad z = u(1+v)w$

39. Use a Jacobian to derive the formulation of triple integrals in cylindrical coordinates.

40.* Starting with the formulation of triple integrals in cylindrical coordinates and by finding a coordinate transformation T from spherical to cylindrical coordinates along with its Jacobian, provide another derivation of the formula for triple integrals in spherical coordinates. (**Hint:** See Example 3 for guidance.)

41. Verify that for the change of variables in Example 4, $|\partial(x,y,z)/\partial(u,v,w)| = 2$ and thus,

$$\iiint_R \left(\frac{3x-z}{2}+\frac{y}{3}\right)dV = \int_{w=0}^{w=1}\int_{v=0}^{v=2}\int_{u=0}^{u=3}\left(\frac{u}{2}+w\right)(2)\,du\,dv\,dw$$
$$=15.$$

42–44 Use a change of variables to evaluate the given integral over the solid R.

42. Evaluate $\iiint_R \frac{4y+2x-z}{4}\,dV$, where R is the solid bounded by the planes $x = 0$, $x = 2$, $z = 0$, $z = 3$, $4y = z$, and $4y = z + 6$.

43. Evaluate $\iiint_R \left(\frac{2x-y}{2}+\frac{3x+5z}{6}\right)dV$, where R is the solid bounded by the planes $z = 0$, $z = 2$, $y = 2x$, $y = 2x - 8$, and $z = 3x$, $z = 3x - 6$.

44. Evaluate $\iiint_R x(2z-y)e^{y/2}\,dV$, where R is the solid bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $y = 2z$, and $y = 2z - 1$.

45. Use a change of variables to provide a second solution to Exercise 47 of Section 14.2. (**Hint:** Using the notation of Exercise 29, notice that after transforming variables according to $x = u$, $y = 2v$, you will be able to integrate on a disk. Perform another change of variables to polar coordinates to finish the problem.)

46. Following the hint given in Exercise 45, solve Exercise 49 of Section 14.2 by changing variables twice.

47. Use the approach of Exercise 45 to derive the formula for the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

48.* After determining the second moment of the ellipsoid of Exercise 47 about the z-axis, show that its radius of gyration about the same is $r_z=\sqrt{\frac{1}{5}\left(a^2+b^2\right)}$. Can you find formulas for r_x and r_y?