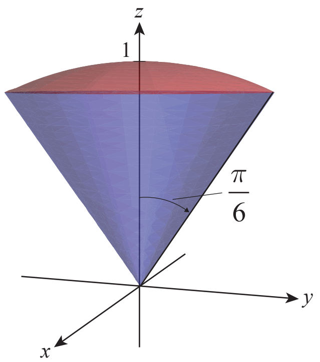


Figure 11

Example 6 Using a Triple Integral in Spherical Coordinates to Find the Volume of a Solid

Find the volume of the solid, shown in Figure 11, bounded below by the cone $\varphi=\pi/6$ and above by the sphere $\rho=1$.

Solution

The lower and upper limits on ρ are 0 and 1, respectively, while the φ-interval is $[0,\pi/6]$ and the θ-interval is $[0,2\pi]$. Since we seek only the volume, we integrate the constant function 1 over S. However, don't forget to include the factor of $\rho^2\sin\varphi$, which is part of the volume differential dV.

$$V=\iiint_S dV=\int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/6}\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=1}\sin\varphi\,d\varphi\,d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/6}\sin\varphi\,d\varphi\,d\theta=\frac{1}{3}\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\pi/6}d\theta$$

$$=\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right)\int_0^{2\pi}d\theta=\frac{\left(2-\sqrt{3}\right)\pi}{3}$$

14.5 Exercises

1–4 Find a set of cylindrical coordinates for the point given in Cartesian coordinates.

1. $(1,1,2)$ **2.** $\left(1,-\sqrt{3},-1\right)$

3. $\left(-6\sqrt{3},-6,0\right)$ **4.** $(-3,-4,-5)$

5–8 Find the Cartesian coordinates of the point given in cylindrical coordinates.

5. $\left(2,\frac{3\pi}{4},\sqrt{2}\right)$ **6.** $(3,\pi,3)$

7. $\left(-1,\frac{\pi}{2},1\right)$ **8.** $\left(4,-\frac{\pi}{3},\frac{\pi}{2}\right)$

9–16 Write the equation in cylindrical coordinates.

9. $x^2+y^2=4x$ **10.** $x^2=2z-y^2$

11. $x^2+(y-1)^2=1$ **12.** $x^2+y^2+z^2=1$

13. $\sqrt{x^2+y^2}=3z$ **14.** $z=x-2y+1$

15. $e^{-\frac{x^2+y^2}{2}}=z$ **16.** $(x-3)^2+y^2=z+9$

17–24 Describe the graph of the equation in words, and change the equation to Cartesian coordinates.

17. $r=1$ **18.** $\theta=\frac{\pi}{6}$

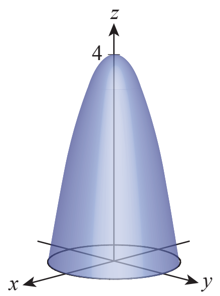
19. $2r^2=2-z^2$ **20.** $2r^2=2-z$

21. $r^2=z^2$ **22.** $r=4\cos\theta$

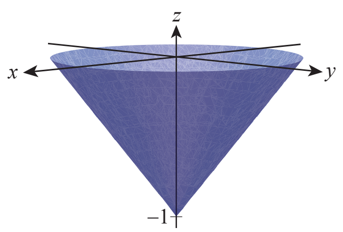
23. $z=\sqrt{1-r^2}$ **24.** $r=4\sec\theta$

25–36 Set up a triple integral in cylindrical coordinates for the volume of the solid S. Do not evaluate the integral.

25. S: The solid bounded by $z=4-\left(x^2+y^2\right)$ and the xy-plane

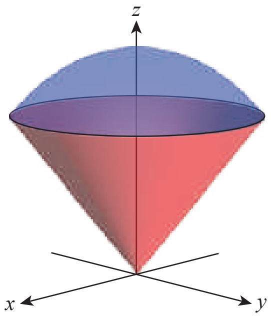


26. S: The solid bounded by $z=\sqrt{x^2+y^2}-1$ and the xy-plane



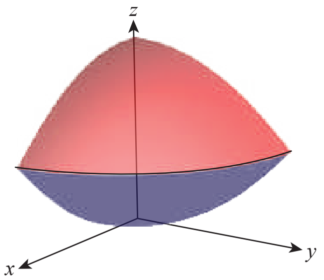
27. S: The solid bounded by $z = 4 - \sqrt{x^2 + y^2}$ and the xy-plane

28. S: The solid bounded above by $z = \sqrt{2 - x^2 - y^2}$ and below by $z = \sqrt{x^2 + y^2}$



29. S: The solid inside both $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 8$

30. S: The solid inside both $z = x^2 + y^2$ and $z = 18 - (x^2 + y^2)$ in the first quadrant



31. S: The solid bounded by $z^2 = 2 + x^2 + y^2$, $z = \sqrt{2}$, and $z = \sqrt{3}$

32.* S: The solid bounded by $z = x^2 + y^2$, $z = 1$, and $z = 4$ (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

33.* S: The solid bounded by $z = 2\sqrt{x^2 + y^2}$, $z = 1$, and $z = 5$ (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

34.* S: The solid bounded by $z^2 + 2 = x^2 + y^2$, $z = -3$, and $z = 3$ (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

35. S: The solid of Exercise 31 (**Hint:** Integrate in the order $dr\,d\theta\,dz$.)

36. S: The solid bounded by $z = e^{x^2+y^2}$ and $z = e^{2-x^2-y^2}$

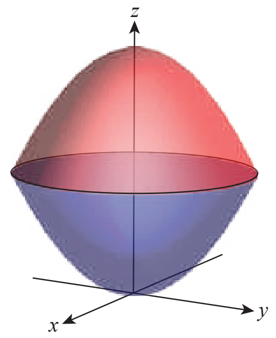
37. Show that if the solid of Example 2 has constant density, the third coordinate of its center of mass is $\bar{z} = \frac{2}{3}$.

38. Use cylindrical coordinates to verify the formula for the volume of a right circular cone of radius R and height h, $V = \frac{1}{3}\pi R^2 h$.

39–53 Use the cylindrical coordinate system to determine the mass of the solid S with the given density function.

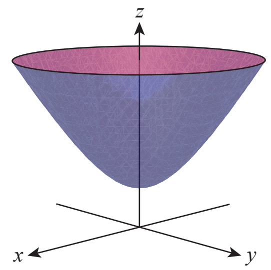
39. S is the solid bounded by $z = 1 - x^2 - y^2$ and the xy-plane, with density function $\rho(x, y, z) = z$.

40. S is the solid bounded by $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$, with constant density ρ.



41. S is the solid upper hemisphere of radius 1 centered at the origin, with its density at the point (x, y, z) being proportional to its distance from the base. (See Exercise 43 of Section 14.4.)

42. S is the solid bounded by the upper sheet of the hyperboloid $z^2 - x^2 - y^2 = 1$ and the plane $z = \sqrt{5}$, with constant density ρ.



43. S is the solid bounded by $z = e^{\sqrt{x^2+y^2}}$ and the plane $z = e$, with density function $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

44. S is the solid bounded by $x^2 + y^2 = 1$, the xy-plane, and $z = e$, with density function $\rho(x, y, z) = e^{-x^2-y^2}$.

45. S is the solid outside $z = 1 - \sqrt{x^2 + y^2}$, bounded by the xy-plane, $z = 1$, and $x^2 + y^2 = 1$, with density function $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

46. S is the solid of Exercise 28, with constant density ρ.

47. S is the solid of Exercise 28, with density function $\rho(x, y, z) = z$.

48. S is the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, with density function $\rho(x, y, z) = e^{\sqrt{x^2+y^2}}$.

49. S is the solid bounded by the cylinder $x^2+(y-1)^2=1$, the xy-plane, and the paraboloid $z=x^2+y^2$; its density at any point (x,y,z) is proportional to the square of the point's distance from the z-axis.

50. S is the solid bounded by the cylinder $(x-1)^2+z^2=1$, the xz-plane, and the paraboloid $y=x^2+z^2$, with density function $\rho(x,y,z)=x$. (**Hint:** Integrate with respect to y first; then define and use the polar coordinates $x=r\cos\theta$ and $z=r\sin\theta$.)

51. S is the solid bounded by $z=\sqrt{x^2+y^2}$ and $z=\left(x^2+y^2\right)^{3/2}$, with density function $\rho(x,y,z)=\sqrt{x^2+y^2}\,e^z$.

52. S is the solid bounded by $z=1/\sqrt{x^2+y^2}$, $2\left(x^2+y^2\right)=1$, $x^2+y^2=1$, and the xy-plane, with density function $\rho(x,y,z)=\sqrt{x^2+y^2}\,e^{\sqrt{x^2+y^2}}$.

53. S is a solid sphere of radius 2, with a cylindrical hole of radius 1 drilled into it along one of its diameters. Its density at any of its points is equal to the square of the distance from the origin.

54–64 Use the cylindrical coordinate system to find the center of mass of the solid S with the given density function.

54. S: The solid bounded by the paraboloids $z=x^2+y^2$ and $z=3-2x^2-2y^2$, with constant density

55. S: The solid inside $x^2+y^2=4$, outside $x^2+y^2=1$, and bounded by the paraboloid $z=16-x^2-y^2$ and the xy-plane, with constant density

56. S: The upper hemisphere of radius R centered at the origin, with constant density

57. S: The upper hemisphere of radius 1 centered at the origin, if its density is proportional to the square of the distance from the origin

58. S: The hemisphere of Exercise 56, if its density is proportional to the distance from the z-axis

59. S: The right circular cylinder of radius 1 and height 2, its base centered at the origin in the xy-plane, with density function $\rho(x,y,z)=e^{-z}$

60. S: The solid inside the paraboloid $z=6-x^2-y^2$ and outside the sphere $x^2+y^2+z^2=8$, with constant density

61. S: The right circular cone of radius R and height h with constant density ρ

62. S: The cone of Exercise 61 with the density of the cone being proportional to the distance from the cone's axis of symmetry

63. S: The cone of Exercise 61 with the density being proportional to the distance from the base

64.* S: The first octant of the unit sphere centered at the origin, its density at any point is proportional to the product of its distances from the coordinate planes (See Exercise 60 of Section 14.4.)

65.* Use cylindrical coordinates to give a second solution to Exercise 61 of Section 14.4. (See the hint given in Exercise 50.)

66–68 Use cylindrical coordinates to find the first and/or second moments of the solid, as indicated.

66. Find the second moment about the z-axis and the corresponding radius of gyration for the solid of Exercise 53. (Suppose it is centered at the origin, with the "hole" in vertical position.)

67. Find the moment of inertia about the z-axis and the corresponding radius of gyration for the cylinder $x^2+y^2=1$, bounded by $z=0$ and $z=1$, if it has constant density ρ.

68. Repeat Exercise 67 for the solid of Exercise 55.

69–76 Find the indicated quantities for the solid from an earlier exercise in this section.

69. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 39

70. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 40

71. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 41

72. The center of mass, second moment I_z, and radius of gyration r_z for the solid of Exercise 43

73. The second moment I_z and radius of gyration r_z for the solid of Exercise 55

74.* The second moment I_z and radius of gyration r_z for the solid of Exercise 59

75. The second moment I_z and radius of gyration r_z for the solid of Exercise 42, with density being proportional to the distance from the xy-plane

76. The second moment I_z and radius of gyration r_z for the solid of Exercise 60

77–80 Find a set of spherical coordinates for the point given in Cartesian coordinates.

77. $\left(\sqrt{3},1,2\right)$

78. $(1,0,1)$

79. $\left(0,-1,-\sqrt{3}\right)$

80. $\left(-2,2,2\sqrt{2}\right)$

81–84 Find the Cartesian coordinates of the point given in spherical coordinates.

81. $\left(\sqrt{2},0,\frac{\pi}{4}\right)$

82. $\left(4,\frac{\pi}{4},\frac{\pi}{3}\right)$

83. $\left(3,-\frac{\pi}{2},\frac{\pi}{2}\right)$

84. $\left(2\sqrt{2},\frac{\pi}{3},\frac{\pi}{4}\right)$

85–92 Change the equation into spherical coordinates.

85. $x^2+y^2+z^2=4$

86. $z=0$

87. $x^2+y^2-z^2=0$

88. $x^2+y^2+\left(z-2\right)^2=4$

89. $y=x$

90. $z=1$

91. $x^2+y^2=4$

92. $x+y+z=1$

93–100 Describe the graph of the equation in words, and change the equation to Cartesian coordinates.

93. $\rho=1$

94. $\theta=\frac{\pi}{3}$

95. $\varphi=\frac{\pi}{4}$

96. $\varphi=\frac{\pi}{6}$

97. $\varphi=\frac{\pi}{2}$

98. $\rho=2\sec\varphi$

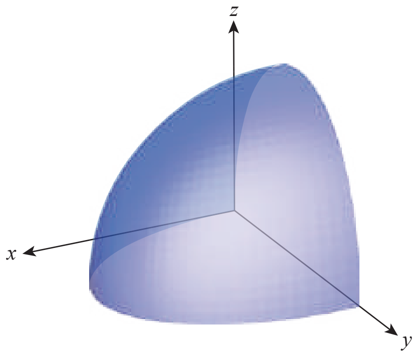
99. $\rho^2=9\csc^2\varphi$

100. $\tan^2\varphi=1$

101–105 Set up a triple integral in spherical coordinates for the volume of the solid *S*. Do not evaluate the integral.

101. *S*: The upper hemisphere of the sphere with radius 2, centered at the origin

102. *S*: The first-octant portion of the sphere $x^2+\left(y-1\right)^2+z^2=1$



103. *S*: The first-octant portion of the sphere $x^2+y^2+\left(z-3\right)^2=9$

104. The "ice-cream cone" of Exercise 45 of Section 14.3, bounded by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=8$

105. The first-octant portion of the "wedge" of the sphere $x^2+y^2+z^2=25$, bounded by the planes $y=\sqrt{3}x$ and $x=\sqrt{3}y$

106–131 Use spherical coordinates to solve the exercise.

106. Evaluate the integral you set up in Exercise 101.

107. Find the volume of the "wedge" of Exercise 105.

108. Derive the formula for the volume of a sphere of radius R, $V=\frac{4}{3}\pi R^3$.

109. Derive the formula for the volume of a right circular cone of radius r and height h, $V=\frac{1}{3}\pi r^2 h$. (**Hint:** Place an inverted cone in the coordinate system with its vertex at the origin and its axis coinciding with the positive z-axis.)

110. Find the mass of the half ball of Exercise 101 if its density is proportional to the distance from its center.

111. Find the volume of the "ice-cream cone" of Exercise 104. (Compare with Exercise 45 of Section 14.3).

112. Find the volume of the solid below the cone $z=\sqrt{x^2+y^2}$ and inside the unit hemisphere $z=\sqrt{1-x^2-y^2}$.

113. Repeat Exercise 112 for the solid below the cone $z=\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+\left(z-2\right)^2=4$, $z\geq 0$.

114. Use Exercise 112 to find the rectangular equation of the cone that divides the upper unit hemisphere into two parts of equal volume.

115. Determine the centroid of the half ball of Exercise 56.

116. Find the centroid of the solid of Exercise 104, assuming it has constant density.

117. Find **a.** the mass and **b.** the centroid of the "ice-cream cone" that is the solid common to the sphere $x^2+y^2+z^2=4z$ and the cone $x^2+y^2=z^2$, if it has constant density ρ.

118. Find the mass of the upper unit hemisphere centered at the origin if its density at the point (x,y,z) is proportional to the distance from its base.

119.* Generalize Exercises 111 and 116 (along with Example 6) by showing that, assuming constant density, the mass and centroid of the "ice-cream cone" bounded by the cone $\varphi=\alpha$ and the sphere of radius R are, respectively, $M=\frac{2}{3}\pi R^3(1-\cos\alpha)\rho$ and $\left(0,0,\frac{3}{8}R(1+\cos\alpha)\right)$.

120. Find the moment of inertia about the z-axis of the solid of Exercise 104. Assume constant density.

121. Find the second moment I_z and radius of gyration r_z of a solid ball of radius R and constant density ρ, if it is centered at the origin.

122. Find I_z and r_z for the solid of Exercise 112.

123. Find the moment of inertia about one of the coordinate axes of the spherical shell bounded by the spheres $x^2+y^2+z^2=2$ and $x^2+y^2+z^2=3$, if its density at any point is proportional to the distance from the origin. (**Hint:** Use the symmetry of the solid.)

124. Find a second solution (one that utilizes spherical coordinates) to Exercise 61.

125.* Find a second solution (one that utilizes spherical coordinates) to Exercise 62.

126.* Find a second solution (one that utilizes spherical coordinates) to Exercise 63.

127. **a.** Describe the solid of integration; then find **b.** its centroid, **c.** its moments of inertia about the coordinate axes, and **d.** its radii of gyration (assuming constant density).

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

128. **a.** Describe the solid of integration and **b.** find its volume by evaluating the integral.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

129.* **a.** Describe the solid of integration and **b.** find its volume by evaluating the integral.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

130. Evaluate $\iiint\limits_S \sin\left(x^2+y^2+z^2\right)^{3/2}dV$, where S is the unit ball centered at the origin.

131. Evaluate $\iiint\limits_S \frac{dV}{\sqrt{x^2+y^2+z^2}}$, where S is the first-octant portion of the solid inside the unit sphere and between the cones $\varphi=\pi/6$ and $\varphi=\pi/3$.

14.5 Technology Exercises

132. Set up an integral for the mass of the solid in Example 3 in Cartesian coordinates, integrating in the order of $dz\,dy\,dx$, and use a computer algebra system to evaluate it.

133. Repeat Exercise 132, but this time use spherical coordinates.

134. Find the mass and center of mass of the hemisphere of Exercise 56, if its density is proportional to the square of the distance from the origin.

135. Use spherical coordinates to find the center of mass of the first octant of the unit sphere centered at the origin, if its density at any point is proportional to the product of the point's distances from the coordinate planes.

136–143 Use a computer algebra system to evaluate the integral in the indicated exercise.

136. Exercise 53

137. Exercise 58

138. Exercise 72

139. Exercise 76

140. Exercise 124

141. Exercise 125

142. Exercise 126

143. Exercise 129