$$I_z = \iiint\limits_S \left(x^2+y^2\right)\rho(x,y,z)\,dV = \rho\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\int_{x^2+3y^2+2}^{6-x^2-y^2}\left(x^2+y^2\right)dz\,dx\,dy$$

$$= \rho\int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}}\left(4x^2-2x^4+4y^2-6x^2y^2-4y^4\right)dx\,dy$$

$$= -\frac{16\rho}{15}\int_{-1}^{1}\left(3y^4-y^2-2\right)\sqrt{2-2y^2}\,dy \qquad y=\sin\theta,\ dy=\cos\theta\,d\theta$$

$$= -\frac{16\rho\sqrt{2}}{15}\int_{-\pi/2}^{\pi/2}\left(3\sin^4\theta-\sin^2\theta-2\right)\cos^2\theta\,d\theta \qquad \text{See Section 7.3.}$$

$$= \sqrt{2}\pi\rho$$

This gives us the radius of gyration about the z-axis.

$$r_z = \sqrt{\frac{I_z}{M}} = \frac{1}{\sqrt{2}}$$

14.4 **Exercises**

1. Verify that

$$\iiint\limits_S \left(3x^2y-xyz^3\right)dV = \int_1^3\int_0^2\int_{-1}^{2}\left(3x^2y-xyz^3\right)dy\,dz\,dx$$

yields the same result as that obtained in Example 1.

2–7 Evaluate the triple integral on the rectangular box S. (Choose a convenient order of integration.)

2. $\iiint\limits_S xy^3z\,dV$, where $S=[-1,3]\times[0,1]\times[1,3]$

3. $\iiint\limits_S dV$, where $S=[1,2]\times[3,4]\times[5,6]$

4. $\iiint\limits_S \left(4xy+x^2yz^2\right)dV$, where $S=[0,1]\times[1,2]\times[-1,1]$

5. $\iiint\limits_S \left(y^2z^2-2x^4y\right)dV$, where $S=[-1,1]\times[0,2]\times[-3,0]$

6. $\iiint\limits_S \frac{xy}{z}\,dV$, where $S=[-1,3]\times[0,3]\times[1,e]$

7. $\iiint\limits_S xye^z\,dV$, where $S=[1,2]\times[-1,3]\times[0,\ln 4]$

8–13 Evaluate the iterated integral.

8. $\int_0^1\int_0^{2-z}\int_0^{1-z} xy^2z\,dx\,dy\,dz$

9. $\int_0^1\int_{x^3}^{x}\int_0^{2xz} 5x\,dy\,dz\,dx$

10. $\int_0^3\int_1^2\int_0^{x+3y}(x+y)\,dz\,dy\,dx$

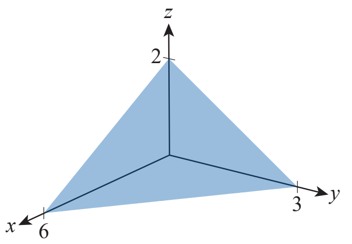
11. $\int_0^1\int_{y+1}^{3y}\int_0^{2xy}(xy)^2\,dz\,dx\,dy$

12. $\int_0^1\int_0^{4-x}\int_0^{4-x-z} dy\,dz\,dx$

13. $\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-y^2}} dz\,dx\,dy$

14–17 Write iterated integrals for $\iiint\limits_S dV$ on the given solid, using the following orders of integration: **a.** $dz\,dy\,dx$, **b.** $dy\,dz\,dx$, and **c.** $dx\,dz\,dy$. Then evaluate one of them to determine the value of the integral.

14. S: The tetrahedron bounded by the coordinate planes and $x+2y+3z=6$



15. S: The cylinder bounded by $x^2+y^2=1$, $z=0$, and $z=2$

16. S: The solid bounded by the parabolic cylinder $z=1-x^2$, $z=0$, $y=0$, and $y=1$

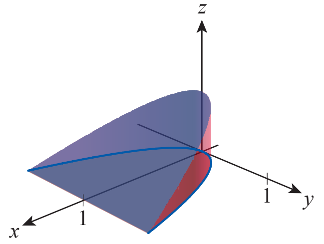
17. S: The solid bounded by $y=x^2$, $y^2=x$, $z=-1$, and $z=1$

18–36 Use a triple integral to find the volume of the solid S.

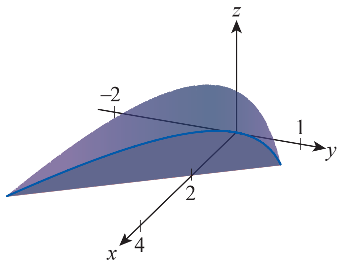
18. S: The tetrahedron bounded by the coordinate planes and the plane $\frac{x}{2}+\frac{y}{5}+\frac{z}{3}=1$

19. Generalize Exercise 18 by finding the volume of the tetrahedron S bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

20. S: The solid bounded by the parabolic cylinder $x = y^2$, and the planes $z = 1 - x$, $z = 0$



21. S: The solid bounded by the parabolic cylinder $x = y^2$, and the planes $x + y + z = 2$ and $z = 0$. Use the order of integration $dz\,dx\,dy$.



22. Revisit Exercise 21, this time integrating in the order $dx\,dz\,dy$.

23. S: The solid bounded by the circular cylinders $x^2 + y^2 = 4$ and $y^2 + z^2 = 4$ (**Hint:** Take advantage of the symmetry of the solid.)

24. S: The solid bounded by the parabolic cylinder $5y = x^2 - 4$ and the planes $y = 1 - z$ and $z = 0$

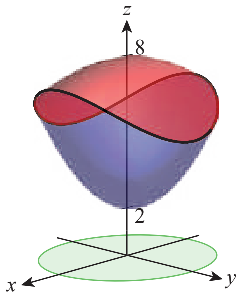
25. S: The solid bounded by the parabolic cylinder $x = 1 - y^2$ and the planes $x = 0$, $z = 0$, and $z = 1 - x$

26. S: The solid bounded by $z = y^2 - 1$, $z = 0$, and $x = \pm 1$

27. S: The solid bounded by $z = x^2$ and the planes $z = 4$, $y = 0$, and $z = y$

28. S: The solid that is common to the paraboloids $z = 9 - 8x^2 - 8y^2$ and $z = x^2 + y^2$ (Exercise 44 of Section 14.3 revisited)

29. S: The solid bounded below by the surface $z = 5x^2 + y^2 + 2$ and above by the surface $z = 8 - x^2 - y^2$. Choose the order of integration $dz\,dy\,dx$. (**Hint:** See Example 2.)



30. Repeat Exercise 29, but this time integrate in the order $dz\,dx\,dy$. (**Hint:** See Example 3.)

31. S: The solid bounded by $z = 2 - x^2 - y^2$, $y = x^2$, $x = y^2$, and $z = 0$

32. S: The solid bounded by $y = z^2$, $y = 2 - z^2$, $x = 0$, and $x = 2$

33. S: The solid bounded by $z = y^2$ and the planes $z = 9 - x$ and $x = 0$

34. S: The solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = x + 6$ (**Hint:** To make your calculations more manageable, integrate with respect to y first, and use the symmetry of the solid.)

35. S: The solid bounded by the elliptic paraboloid $z = 4x^2 + y^2$ and the plane $z = 2y + 3$ (**Hint:** Choose the order of integration carefully and use the symmetry of the solid.)

36. S: The solid bounded by the surfaces $z = 2 - \sqrt{4x^2 + y^2}$ and $z = 0$ (**Hint:** Start integrating with respect to y or x and use the symmetry of the solid.)

37. Write the triple integral over the solid of Exercise 27 in three different ways, in the orders of $dy\,dz\,dx$, $dx\,dz\,dy$, $dx\,dy\,dz$, and evaluate them. (**Note:** Quite possibly, you have already handled one of these integrals in Exercise 27.)

38. Repeat Exercise 37 for the integral in Exercise 33 using the following orders of integration: $dx\,dz\,dy$, $dy\,dz\,dx$, and $dz\,dx\,dy$.

39. Suppose there are one-variable functions g, h, and k such that $f(x,y,z)=g(x)\cdot h(y)\cdot k(z)$ and $S=[p,q]\times[r,s]\times[t,u]$. Prove the following.

$$\iiint_S f(x,y,z)\,dV$$

$$=\left[\int_p^q g(x)\,dx\right]\cdot\left[\int_r^s h(y)\,dy\right]\cdot\left[\int_t^u k(z)\,dz\right]$$

(Note that this is a generalization of Exercise 80 of Section 14.1 to triple integrals.)

40–43 A solid S with variable density is given. Use a triple integral to find its mass.

40. S: The tetrahedron of Example 4, with its density at the point (x,y,z) being proportional to the point's distance from the tetrahedron's base (**Hint:** Integrate over the solid the density function $\rho(x,y,z)=k\cdot z$, where k is a constant.)

41. S: The tetrahedron bounded by $z=4-x-2y$ and the coordinate planes, with its density at the point (x,y,z) being proportional to the square of the distance from the origin (As in the previous exercise, denote the constant of proportionality k.)

42. S: The tetrahedron bounded by $z=3-2x-6y$ and the coordinate planes, with the density at any point being proportional to the sum of its coordinates

43. S: The solid upper hemisphere of radius 1 centered at the origin, with its density at the point (x,y,z) being proportional to the distance from the base

44–47 Just like we did with two-variable functions (see Section 14.2), we can define the **average value** of $f(x,y,z)$ over a solid S as follows.

$$\text{Average value of } f \text{ over } S=\frac{1}{\text{Volume}(S)}\iiint_S f(x,y,z)\,dV$$

Use the above definition to find the average value of f over S.

44. $f(x,y,z)=xyz$;
S: The cube $[0,a]\times[0,a]\times[0,a]$

45. $f(x,y,z)=\dfrac{1}{\sqrt{x}}$;
S: The cube $[0,a]\times[0,a]\times[0,a]$

46. $f(x,y,z)=xy\cos z$;
S: The cube $[0,\pi/2]\times[0,\pi/2]\times[0,\pi/2]$

47.* $f(x,y,z)=xyz$; S: The first-octant region of the sphere $x^2+y^2+z^2=R^2$

48. Finish Example 3 by showing that the value of the integral is $2\sqrt{2}\pi$.

49. **a.** Describe the solid of integration and **b.** find its centroid (assuming constant density).

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

50–62 A solid S with constant or variable density is given. Use a triple integral to find the coordinates of its center of mass.

50. S: The tetrahedron bounded by the coordinate planes and $x+y+z=2$, with constant density

51. S: The tetrahedron of Exercise 18, with constant density

52. S: The solid of Exercise 20, with constant density

53. S: The solid of Exercise 27, with constant density

54. S: The solid of Exercise 26, with constant density

55. S: The solid of Exercise 21, with constant density

56. S: The rectangular prism $[-1,1]\times[-1,1]\times[0,4]$, its density at each point is inversely proportional to the square root of the point's distance from the base

57. S: The cube of Exercise 45, its density at each point is proportional to the square of the point's distance from the origin

58. S: The tetrahedron of Exercise 50, its density at each point is proportional to the square of the point's distance from the origin

59. S: The tetrahedron of Example 4, its density at each point is proportional to the distance from the base

60.* S: The first octant of the unit sphere centered at the origin, its density at any point is proportional to the product of its distances from the coordinate planes

61. S: The solid half cylinder bounded by $z=\sqrt{1-y^2}$, $x=0$, and $x=1$, its density at any point is proportional to the square of the point's distance from the origin

62. S: The tetrahedron of Exercise 19

63. Verify that the second moments and radii of gyration for the solid of Example 4 are as follows.

$$I_x=\frac{8\rho}{15},\quad I_y=I_z=\frac{\rho}{3},\quad r_x=\frac{2}{\sqrt{5}},\quad r_y=r_z=\frac{1}{\sqrt{2}}$$

64–71 Find the center of mass and the radii of gyration of the given solid S. Assume S is made of a substance with constant density ρ.

64. $S=[-1,3]\times[0,1]\times[1,3]$, the rectangular box of Exercise 2

65. S: The cube of Exercise 45

66. S: The solid bounded by $z=4-2x-y$ and the coordinate planes

67. S: The solid bounded by $z=3-2x-6y$ and the coordinate planes

68. S: The solid bounded by $z=y^2$, the planes $z=1-x$ and $x=0$

69. S: The solid bounded by $z=y^2-1$, $z=0$, and $x==1$

70. S: The solid bounded by $z=2-x^2-y^2$, the coordinate planes, and the planes $x=1$ and $y=1$

71. S: The first-octant region of the cylinder $x^2+y^2=1$, bounded by the coordinate planes and $z=1$

72–75 Find the second moments and radii of gyration of the indicated solid.

72. The solid of Exercise 56

73. The solid of Exercise 57

74. The solid of Exercise 59

75. The solid of Exercise 58

14.4 Technology Exercises

76–79 Use a computer algebra system to find the center of mass and the radii of gyration for the given solid with nonconstant density. (Note that even though it does give nice answers, in some cases even a computer algebra system requires a relatively long time for calculating them!)

76. S: The upper hemisphere of radius 1, centered at the origin, the density at any point is inversely proportional to the square root of its distance from the base

77. S: The first octant region of the solid in Exercise 76, the density at any point is proportional to the product of its distances from the coordinate planes

78. S: The solid of Exercise 66, its density at any point being proportional to the square of the point's distance from the origin

79. S: The solid of Exercise 67, its density at any point being proportional to the square root of its distance from the base