

In some cases, the best choice of a coordinate system might not be immediately clear. In our next example, the form of the integrand may lead us to try polar coordinates, but the shape of the region R is better suited for Cartesian coordinates. The integral is actually easier to evaluate with Cartesian coordinates, as you will see in Exercise 1, but for the sake of illustration and comparison we will use polar coordinates.

Example 4 Using a Double Integral in Polar Coordinates to Find Volume of a Solid

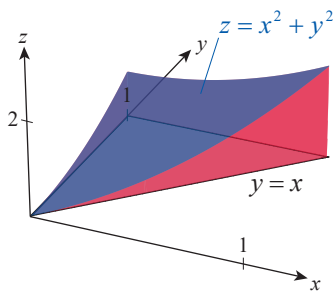


Figure 7

Let R be the triangle in the xy -plane with boundary lines $y = x$, $x = 0$, and $y = 1$. Find the volume of the solid bounded below by the region R and above by the surface $z = x^2 + y^2$.

Solution

The solid as described is pictured in Figure 7, and the region R is shown in Figure 8. The lines $y = x$ and $x = 0$ correspond to the θ -limits $\theta = \pi/4$ and $\theta = \pi/2$, and each radial slice through R has a lower r -limit $r = 0$. Each upper r -limit corresponds to a point on the line $y = 1$, but this must be expressed in the form $r = g_2(\theta)$ for us to integrate using polar coordinates. We make the translation by noting that $y = r \sin \theta$, so

$$y = 1 \Leftrightarrow r \sin \theta = 1 \Leftrightarrow r = \csc \theta.$$

Putting all the pieces together, we obtain the following volume.

$$\begin{aligned} \text{Volume} &= \iint_R (x^2 + y^2) dA = \int_{\pi/4}^{\pi/2} \int_0^{\csc \theta} r^2 r dr d\theta = \frac{1}{4} \int_{\pi/4}^{\pi/2} [r^4]_{r=0}^{r=\csc \theta} d\theta \\ &= \frac{1}{4} \int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta = \frac{1}{4} \int_{\pi/4}^{\pi/2} \csc^2 \theta (1 + \cot^2 \theta) d\theta && \csc^2 \theta = 1 + \cot^2 \theta \\ &= \frac{1}{4} \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta + \frac{1}{4} \int_{\pi/4}^{\pi/2} \csc^2 \theta \cot^2 \theta d\theta && u = \cot \theta \\ &&& du = -\csc^2 \theta d\theta \\ &= \frac{1}{4} [-\cot \theta]_{\pi/4}^{\pi/2} + \frac{1}{4} \left[-\frac{1}{3} \cot^3 \theta \right]_{\pi/4}^{\pi/2} = \frac{1}{3} \end{aligned}$$

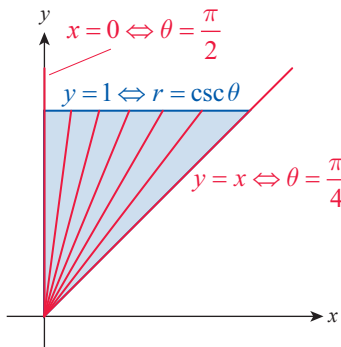


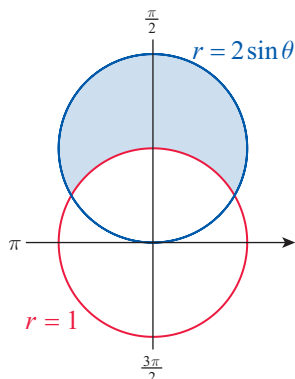
Figure 8

14.3 Exercises

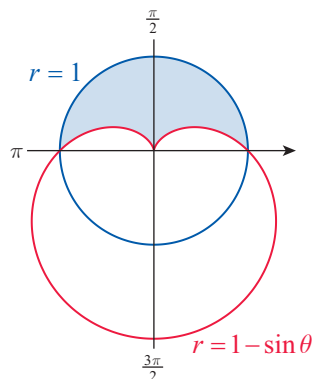
1. Use Cartesian coordinates to determine the volume of the solid in Example 4.

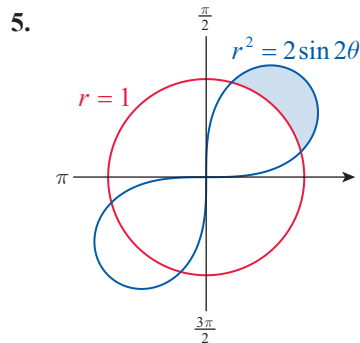
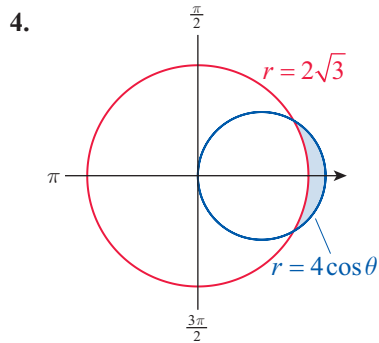
2–5 Use a double integral in polar coordinates to find the area of the shaded region.

2.



3.





6–18 Use a double integral in polar coordinates to find the area of the region R .

6. R : The region inside the circle $r = 2$
7. R : The region inside the circle $r = 6 \sin \theta$
8. R : The region inside the cardioid $r = 1 - \sin \theta$
9. R : The region common to the circles $r = 6 \cos \theta$ and $r = 3$
10. R : The region inside the cardioid $r = 1 + \cos \theta$ and outside the unit circle centered at the origin
11. R : The region inside the circle $r = 2$ and outside the limaçon $r = \frac{3}{2} + \sin \theta$
12. R : The region bounded by the spiral $r = 2\theta$ and the polar axis ($0 \leq \theta \leq \pi$)
13. R : The region inside one petal of the rose $r = 2 \cos 2\theta$ and outside the circle $r = 1$
14. R : The region inside $r = 4 + 2 \sin \theta$ and outside $r = 3$
15. R : The inner loop of the limaçon $1 + 2 \cos \theta$
16. R : The region inside the cardioid $r = 2 - 2 \sin \theta$ and outside the circle $r = 2$
17. R : The region inside the circle $r = 1$ and outside the cardioid $r = 1 + \sin \theta$
18. R : The region inside the circle $r = 2 \sin \theta$, but outside the lemniscate $r^2 = 2 \cos 2\theta$ (**Hint**: Divide R into appropriate subregions and use symmetry.)

19–26 Evaluate the double integral by changing to polar coordinates. (Sketching the region of integration is helpful.)

19. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \frac{dy dx}{x^2 + y^2 + 2}$

20. $\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} dy dx$

21. $\int_{-1}^0 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^{3/2} dx dy$

22. $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \cos(x^2 + y^2) dy dx$

23. $\int_0^2 \int_0^{\sqrt{2y-y^2}} (x^2 + y^2)^{5/2} dx dy$

24. $\int_0^{2a} \int_0^{\sqrt{2ay-y^2}} 2 dx dy$

25. $\iint_R x \sqrt{x^2 + y^2} dA$; R : The region enclosed by the first-quadrant loop of the lemniscate $r^2 = 2 \sin 2\theta$

26. $\iint_R y dA$; R : The region inside the circle $r = 1$ and outside $r = \sin \theta$

27–30 Convert the integral into a Cartesian double integral and evaluate it.

27. $\int_0^{\pi/4} \int_0^{\sec \theta} r^2 \cos \theta dr d\theta$

28. $\int_{\pi/3}^{\pi/2} \int_0^{2 \csc \theta} r^3 \sin \theta \cos \theta dr d\theta$

29. $\int_{\pi/4}^{\arctan 2} \int_0^{\csc \theta} r^3 \sin 2\theta dr d\theta$

30. $\int_0^{\arctan(1/2)} \int_0^{2 \sec \theta} r^5 \sin^2(2\theta) dr d\theta + \int_{\arctan(1/2)}^{\pi/2} \int_0^{\csc \theta} r^5 \sin^2(2\theta) dr d\theta$

31–34 Make your choice between the Cartesian and polar coordinate systems and evaluate the double integral.

31. $\iint_R \sqrt{x^2 + y^2} dA$; $R: x^2 + y^2 \leq 4$

32. $\iint_R (x^2 + y^2) dA$; R : The region bounded by $y = x$, $y = x/2$, and $x = 2$

33. $\iint_R e^{\sqrt{x^2+y^2}} dA$; R : The first-quadrant region of $x^2 + y^2 \leq 1$

34. $\iint_R (x^2 + y^2)^2 dA$; R : The region bounded by $y = 0$, $x = 1$, and $y = x$

35. In Section 9.4, we derived the following formula for the area A of a region bounded by $\theta = a$, $\theta = b$, and the polar equation $r = f(\theta)$.

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

Use a double integral in polar coordinates and the discussion of this section to derive the formula above.

36–44 Use double integration in polar coordinates on an appropriate region to find the volume of the solid S bounded by the given surfaces.

36. S : The solid bounded by the xy -plane and the paraboloid $z = 1 - x^2 - y^2$

37. S : The solid bounded by the xy -plane and the paraboloid $z = 16 - x^2 - y^2$

38. S : The solid bounded by $z = x + 2y + 7$ and the cylinder $x^2 + y^2 = 4$

39. S : The solid bounded by $z = 2 - \sqrt{x^2 + y^2}$ and the xy -plane

40. S : The solid bounded by $z = 3x + 5y + 9$, $z = x + 2y + 3$, and the cylinder $x^2 + y^2 = 2y$

41. S : The solid bounded by $z = x + y + 8$, the xy -plane, and the cylinder $r = 1 + \sin \theta$

42. S : The solid bounded by $z = x^2 + y^2$, the xy -plane, and the cylinder $r = 2 + \cos \theta$

43. S : The solid that is common to the paraboloids $z = 2(x^2 + y^2)$ and $z = 12 - x^2 - y^2$ (**Hint**: Note that calculating the curve of intersection of the two surfaces will yield the region of integration.)

44. S : The solid that is common to the paraboloids $z = 9 - 8x^2 - 8y^2$ and $z = x^2 + y^2$ (See the hint given in Exercise 43.)

45. Use double integration in polar coordinates to find the volume of the solid in the shape of an ice-cream cone bounded by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 8$.

46. Use double integration in polar coordinates to derive the formula for the volume of a sphere of radius R , $V = \frac{4}{3}\pi R^3$.

47. Use double integration in polar coordinates to find the volume of the solid inside the paraboloid $z = 6 - x^2 - y^2$ and above the sphere $x^2 + y^2 + z^2 = 8$.

48.* Recall the wooden toy piece from Exercise 17 of Section 6.1. We will generalize that problem as follows. Suppose a cylindrical hole of radius r is drilled through the center of a sphere of radius R . Use double integration in polar coordinates to show that the volume of the remaining ringlike solid is $V = \frac{4}{3}\pi(R^2 - r^2)^{3/2}$.

49. Assuming constant density, use double integration in polar coordinates to find the center of mass of the region outside the circle $r = 1$ and inside the cardioid $r = 1 + \sin \theta$. (**Hint**: Use the symmetry of the region.)

50. Use double integration in polar coordinates to find the center of mass of the region outside the cardioid $r = 2 - 2 \sin \theta$ and inside the circle $r = 2 \cos \theta$. As in Exercise 49, we assume constant density.

51. Determine the second moments I_x , I_y , and I_0 for the thin plate of constant density ρ inside the circle $r = 4 \cos \theta$ and outside the circle $r = 2$.

52. Determine I_x , I_y , I_0 , and the corresponding radii of gyration for the quarter annulus $\{(x, y) \mid 0 \leq x \leq 2, \sqrt{1-x^2} \leq y \leq \sqrt{4-x^2}\}$. Assume the annulus has constant density ρ .

53. Use double integration in polar coordinates to find I_x , I_y , I_0 , and the corresponding radii of gyration for the disk of Example 4 of Section 14.2, if the disk is shifted upward by a units.

54. Use double integration in polar coordinates to find I_x , I_y , and I_0 directly for the annulus in Exercise 60 of Section 14.2.

55. Assuming it has constant density 1, find I_x , I_y , I_0 , and the corresponding radii of gyration for the first-quadrant loop of the lemniscate $r^2 = \sin 2\theta$.

56.* Repeat Exercise 55 for the region inside the circle $r = 2$ and outside $r = 2 \cos(\theta/2)$. (**Hint**: Take advantage of the symmetry of the region.)

57. By changing to polar coordinates, verify the value of the following improper double integral.

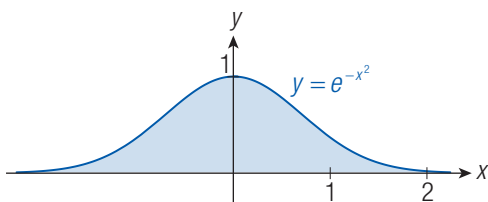
$$\int_0^{\infty} \int_0^{\infty} \frac{dy dx}{(1+x^2+y^2)^2} = \frac{\pi}{4}$$

(**Hint:** Integrate on the first-quadrant region of a disk of radius r , and let $r \rightarrow \infty$.)

- 58–59 In the next two exercises, you will be guided to use double integrals in polar coordinates to find the area under the bell curve $y = e^{-x^2}$, that is, to prove that

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

(This is an important integral not only in mathematics, but also in statistics, engineering, and physics.)



58. Use the technique of Exercise 57 to show

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \pi.$$

59. Evaluating J in rectangular coordinates over the square $[-a, a] \times [-a, a]$ (and letting $a \rightarrow \infty$), use Exercise 80 of Section 14.1 to show that $I^2 = J$. Conclude that $I = \sqrt{\pi}$.

Concept Check

60–63 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

60. If R is a square, and $f(r, \theta)$ is defined on R , then it is impossible to evaluate $\iint_R f(r, \theta) dA$ using polar coordinates.
61. If R is a bounded region, and $f(r, \theta) = 1$ on R , then the area A of R can be found as an iterated integral $\iint_R f(r, \theta) dr d\theta$.
62. The decision regarding which coordinate system (i.e., rectangular or polar) to use when evaluating $\iint_R f dA$, hinges upon the geometry (i.e., shape) of the region R .
63. Polar coordinates are suitable to determine the radii of gyration for certain planar regions.

14.3 Technology Exercises

64–66 Use a computer algebra system to evaluate the integral in both the rectangular and polar coordinate systems. Compare your answers.

64. $\int_0^1 \int_0^x \sqrt{x^2 + y^2} dy dx$

65. $\int_0^1 \int_{1-y}^{2-2y} xy dx dy$

66. $\int_0^1 \int_y^{\sqrt{2-y^2}} \sqrt{x^2 + y^2} dx dy$

(Note that even a CAS has a relatively hard time evaluating the integral of Exercise 66, while it is easy to determine, even just by a paper-and-pencil calculation, after converting it to polar coordinates!)

67–69 Assume that the thin plate covering the given region has a constant density of 1. Use a computer algebra system to find the moments of inertia I_x, I_y, I_0 , and the corresponding radii of gyration.

67. The region bounded by $r = 1 - \sin \theta$
68. The region bounded by both $r = 1$ and $r = 1 + \sin \theta$
69. The region bounded by the inner loop of the limaçon $r = 1 + 2 \cos \theta$