

## 14.2 Exercises

**1–12** Use double integration to find the area of the region bounded by the graphs of the given equations.

- $y = 2x, \quad y^2 = 4x^3$
- $\sqrt{2}y = x^2, \quad x^2 + y^2 = 4, \quad x = 0$
- $x^2 = y, \quad x = y^2$
- $y^2 = 4x, \quad 2x = 4 - y$
- $y^2 = 9x, \quad y^2 = 10 - x$
- $y = x, \quad xy = 36, \quad y = 12$
- $2x = 8 - y, \quad y = 6x, \quad 2y = x^3$
- $y = 2\sqrt[4]{x}, \quad y = x, \quad y = 2$
- $y = x + 10, \quad y = 8x - x^2$
- $x = (y - 2)^2, \quad x = (y + 2)^2, \quad x = 0$
- $y = \frac{x}{4}, \quad y = 4x, \quad xy = 1$
- $y = \frac{x^2}{2}, \quad y = \frac{1}{1 + x^2}$

**13–24** Find the average value of  $f(x, y)$  over the region bounded by the graphs of the given equations.

- $f(x, y) = x^2 - y; \quad y = 2x, \quad y^2 = x^3$
- $f(x, y) = 2y, \quad y = \sqrt{3}x; \quad x^2 + y^2 = 1, \quad x = 0$
- $f(x, y) = -x; \quad y = 2x^2, \quad 4x = y^2$
- $f(x, y) = \frac{y^2}{2}; \quad x = y^2, \quad x = 2 - y$
- $f(x, y) = x^2y; \quad x^2 = 4y, \quad x^2 = 5 - y$
- $f(x, y) = xy; \quad y = x, \quad xy = 1, \quad y = 0, \quad x = 2$
- $f(x, y) = 5y^2; \quad x = 2 - y, \quad y = x^3, \quad x = 0$
- $f(x, y) = xy^3; \quad y^3 = x, \quad y = x, \quad x \geq 0$
- $f(x, y) = \sqrt{x}; \quad y = x + 4, \quad y = 6x - x^2$
- $f(x, y) = e^{x+1}; \quad y = (x - 1)^2, \quad y = (x + 1)^2, \quad y = 0$
- $f(x, y) = xy; \quad y = \frac{x}{2}, \quad y = 2x, \quad xy = 2$
- $f(x, y) = x^2; \quad y = \frac{x^2}{4}, \quad y = \frac{1}{2 + 2x^2}$

**25.** Find the center of mass of the triangular plate of Example 3 if the density of the plate  $\rho(x, y) = c$  is a constant. Compare your answer with that given in Example 3.

**26–35** Find the center of mass of the plane region of varying density that is bounded by the graphs of the given equations.

- $2y = 4 - x, \quad x = 0, \quad y = 0; \quad \rho(x, y) = xy$
- $y = 2x, \quad y = 3 - x, \quad y = 0; \quad \rho(x, y) = x + y$
- $y = x^{3/2}, \quad y = \sqrt{x}; \quad \rho(x, y) = \sqrt{x}$
- $y = \sqrt{1 - x^2}, \quad x = 0, \quad y = 0; \quad \rho(x, y) = x + 2y$
- $y = 4 - x^2, \quad y = 2 - x; \quad \rho(x, y) = x^2y$
- $y = \frac{1}{x^3}, \quad x = \frac{1}{2}, \quad x = 1, \quad y = 0; \quad \rho(x, y) = 2x^3 + y$
- $y = x, \quad y = \sqrt{x}; \quad \rho(x, y) = y - x$
- $xy^2 = 3, \quad y = \frac{1}{3}, \quad y = 1; \quad \rho(x, y) = xy$
- $y = \sqrt{x}, \quad xy = 1, \quad y = 0, \quad x = 2; \quad \rho(x, y) = x^2$
- $y = e^x, \quad x = 0, \quad y = 0, \quad x = 1; \quad \rho(x, y) = xy$

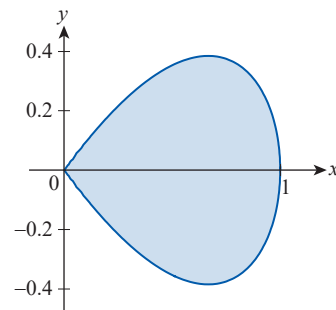
**36–50** Determine  $I_x$ ,  $I_y$ , and  $I_0$  for the thin plate of constant density  $\rho$  modeled by the planar region  $R$ . Then find its corresponding radii of gyration.

- $R$ : The region bounded by the graph of  $y = 1 - x^2$  and the  $x$ -axis
- $R$ : The square  $[-a, a] \times [-a, a]$
- $R$ : The square of Exercise 37 shifted in the positive  $x$ -direction by  $a$  units
- $R$ : The rectangle  $[-l/2, l/2] \times [-w/2, w/2]$
- $R$ : The rectangle of Exercise 39 shifted in the positive  $y$ -direction by  $w/2$  units
- $R$ : An equilateral triangle of side length  $2a$ , with its center at the origin, and a vertex on the positive  $y$ -axis
- $R$ : The triangle of Exercise 41 shifted vertically upward so that its base is on the  $x$ -axis
- $R$ : The square of Exercise 37 with the smaller square  $[-b, b] \times [-b, b]$  removed ( $b < a$ ) (Solve this problem without using the Principle of Superposition.)
- $R$ : The region bounded by the graph of  $y = 2 - \frac{x^2}{2}$  and the  $x$ -axis
- $R$ : The region bounded by the graph of  $y = \cos x$ ,  $-\pi/2 \leq x \leq \pi/2$ , and the  $x$ -axis

46.  $R$ : The region bounded by the graph of  $y = \sin x$ ,  $0 \leq x \leq \pi$ , and the  $x$ -axis
- 47.\*  $R$ : The region bounded by the ellipse  $4x^2 + y^2 = 4$
- 48.\*  $R$ : The region inside the ellipse  $4x^2 + y^2 = 4$  and outside the circle  $x^2 + y^2 = 1$  (Solve this problem without using the Principle of Superposition.)
- 49.\*  $R$ : The region bounded by the ellipse  $16x^2 + 25y^2 = 400$
50.  $R$ : The region bounded by the graphs of  $y = e^x$ ,  $x = \pm 1$ , and the  $x$ -axis

**51–57** As we have seen in Example 5, the Principle of Superposition is a very helpful aid in determining moments of certain objects. Another principle that simplifies the calculation of moments is called the *Parallel Axis Theorem*. It states that if we know the moment  $I_C$  of an object about an axis through its center of mass, then the moment of the same object about a parallel axis is  $I = I_C + Md^2$ , where  $d$  is the distance between the two axes. In these exercises, you will be asked to revisit some of the previous problems, using the Parallel Axis Theorem. To help you get started, we provided a hint in Exercise 51. You should also use the hint's approach with subsequent exercises.

51. Using the answer from Exercise 37, use the Parallel Axis Theorem to obtain a second solution to Exercise 38. (**Hint:** Note that in this case,  $I_x$  doesn't change. Can you see why? As for  $I_y$ , note that after the shift, the new axis is  $a$  units away from the old. After determining  $I_y$  with the help of the Parallel Axis Theorem, using  $d = a$ ,  $I_0$  is readily obtained by adding  $I_x$  and  $I_y$ .)
52. Using the answer from Exercise 39, use the Parallel Axis Theorem to obtain a second solution to Exercise 40.
53. Using the answer from Exercise 41, use the Parallel Axis Theorem to obtain a second solution to Exercise 42.
54. Using the answer from Exercise 45, use the Parallel Axis Theorem to obtain a second solution to Exercise 46.
55. Use the Parallel Axis Theorem to find  $I_x$ ,  $I_y$ ,  $I_0$ , and the corresponding radii of gyration for the ellipse in Exercise 49, if it is shifted to the right so that its left focus coincides with the origin.
56. Use the Parallel Axis Theorem to find  $I_x$ ,  $I_y$ ,  $I_0$ , and the corresponding radii of gyration for the disk in Example 4, if it is shifted upward by  $a$  units (so that it becomes tangential to the  $x$ -axis).
57. Starting with the answer you gave to Exercise 37 (and changing the notation appropriately), use the Parallel Axis Theorem to provide a second solution to Example 6.
58. Use Exercise 37 and the Principle of Superposition to provide a second solution to Exercise 43.
59. Use Exercise 47 and the Principle of Superposition to provide a second solution to Exercise 48.
60. Consider an annulus, as in Example 5, with  $a_2 = 1$ ,  $a_1 = \frac{1}{2}$ , and  $\rho = 1$ . Furthermore, suppose that it is centered at  $(a_1, 0)$ . Starting with the results found for a disk in Example 4, use a combination of the Principle of Superposition and the Parallel Axis Theorem to find  $I_x$ ,  $I_y$ , and  $I_0$  for this annulus.
61. Suppose that  $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$  is removed from the 2-by-2 square  $[-1, 1] \times [-1, 1]$  and that its center is shifted to the point  $(0, \frac{1}{2})$ . Starting with the results of Example 6 (or Exercise 37) and assuming that the density is 1, use a combination of the Principle of Superposition and the Parallel Axis Theorem to find  $I_x$ ,  $I_y$ , and  $I_0$ .
- 62–67** Consider the thin plates from the indicated problems with the given nonconstant densities. Determine  $I_x$ ,  $I_y$ ,  $I_0$ , and the corresponding radii of gyration.
62. Example 6;  $\rho(x, y) = xy$
63. Exercise 36;  $\rho(x, y) = y$
64. Exercise 45;  $\rho(x, y) = y$
65. Exercise 46;  $\rho(x, y) = x$
- 66.\* Exercise 47;  $\rho(x, y) = x^2$
67. Exercise 50;  $\rho(x, y) = y$
- 68.\* Determine  $I_x$ ,  $I_y$ ,  $I_0$ , and the corresponding radii of gyration for the thin plate of constant density  $\rho$  enclosed by the loop  $y^2 = (1-x)x^2$ .



69. Suppose a thin, but solid, disk of uniform density (such as in Example 4) is rolling down an incline of height  $h$  (without slipping) in a race against the annulus of Example 5 and the square of Example 6, which is greased so it is sliding without friction (you can think of the latter as a sliding box). Which object will win, and in what order are they going to arrive at the bottom of the incline? Explain. (**Hint:** Use the fact that the initial potential energy of the object is eventually going to be shared between kinetic and rotational energies.)

## Concept Check

**70–73** Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

70. If  $C$  is the average value of  $f(x, y)$  over the region  $R$ , then  $\iint_R C \, dA = \iint_R f(x, y) \, dA$ .
71. If we increase the constant density of a planar object (or solid) from  $\rho$  to  $2\rho$ , the coordinates  $\bar{x}$ ,  $\bar{y}$ , and  $\bar{z}$  of its center of mass will double.
72. If we increase the constant density of a planar object (or solid) from  $\rho$  to  $2\rho$ , then its second moments will double.
73. If we increase the constant density of a planar object (or solid) from  $\rho$  to  $2\rho$ , then its radii of gyration will double.

## 14.2 Technology Exercises

**74–76** Write a program for a computer algebra system or programmable calculator that uses the integral definition to find the moments of inertia  $I_x$ ,  $I_y$ ,  $I_0$ , and the corresponding radii of gyration. Use it to check the answers you obtained previously in the given exercise.

74. Exercise 55

75. Exercise 56

76. Exercise 59

77. Find the average value of  $f(x, y) = \sqrt{y}$  on the region bounded by the graphs of  $y = (x-1)^2$ ,  $y = (x+1)^2$  and  $y = 0$ .