

14.1 Exercises

- Suppose $f(x, y) = x^2$ is defined on the square $R = [0, 4] \times [0, 4]$. Estimate $\iint_R f(x, y) dA$ using the Riemann sum approximation corresponding to $n = m = 4$, with each sample point (x_{ij}^*, y_{ij}^*) chosen to be the center point of the respective subsquare, $1 \leq i, j \leq 4$.
- Use Fubini's Theorem to evaluate the integral of Exercise 1 and compare its true value to your estimate.
- Evaluate the iterated integral and verify that your answer is equal to that obtained in Example 3.

$$\int_1^{5/2} \int_{1/2}^2 (15 - x^2 - 2y^2) dy dx$$

4–15 Evaluate the iterated integral.

- $\int_0^1 \int_0^4 (3x - 2y) dx dy$
- $\int_0^3 \int_1^2 xy^2 dy dx$
- $\int_2^4 \int_{-1}^3 (5x + xy - 3y) dx dy$
- $\int_{-2}^2 \int_0^1 (x^3 y - 9) dy dx$
- $\int_{-1}^0 \int_0^3 (xy^4 - x^4 y) dx dy$
- $\int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \sin y \cos x dx dy$
- $\int_1^2 \int_0^1 y^2 e^{-x} dy dx$
- $\int_{-3/2}^0 \int_0^{\pi} (x^2 y - \cos x) dx dy$
- $\int_1^2 \int_1^2 \left(\frac{x}{y} - \frac{y}{x} \right) dx dy$
- $\int_{1/2}^2 \int_1^e \frac{1}{x^2 y} dy dx$
- $\int_2^3 \int_0^1 \frac{x}{\sqrt{1-y^2}} dy dx$
- $\int_0^1 \int_1^e \frac{1}{x\sqrt{y}} dx dy$

16–19 Suppose $\iint_{R_1} f(x, y) dA = 4$, $\iint_{R_1} g(x, y) dA = 5$, $\iint_{R_2} g(x, y) dA = -1$, and $\iint_{R_1 \cup R_2} f(x, y) dA = 12$ on the rectangular regions $R_1 = [-1, 3] \times [2, 4]$ and $R_2 = [-5, 5] \times [-2, 2]$. Use the properties of double integrals to evaluate the integral.

- $\iint_{R_1} [2f(x, y) + 3] dA$
- $\iint_{R_1} [4f(x, y) - g(x, y)] dA$
- $\iint_{R_1 \cup R_2} [3f(x, y) + 5g(x, y)] dA$
- $\iint_{R_2} [3f(x, y) - 2g(x, y)] dA$

20–22 Evaluate the given integral on the rectangular region $R = [-1, 3] \times [0, 4]$. (**Hint:** Recall that $\llbracket x \rrbracket$ denotes the greatest integer less than or equal to x .)

- $\iint_R (\llbracket y \rrbracket - \llbracket x \rrbracket) dA$
- $\iint_R \llbracket x \rrbracket \cdot \llbracket y \rrbracket dA$
- $\iint_R x \cdot \llbracket y \rrbracket dA$

23–34 Evaluate the iterated integral.

- $\int_0^1 \int_{x^2}^x (x^2 + y^2) dy dx$
- $\int_0^{1/2} \int_0^y (x + y)^2 dx dy$
- $\int_0^2 \int_{x/2}^{3-x} xy dy dx$
- $\int_0^3 \int_{y/3}^{7-2y} x^2 y dx dy$
- $\int_1^e \int_0^{\sqrt{9-x^2}} \frac{2y}{x} dy dx$
- $\int_0^3 \int_0^{\sqrt{9-x^2}} \frac{1}{\sqrt{9-x^2}} dy dx$
- $\int_0^1 \int_y^y e^y dx dy$
- $\int_0^1 \int_y^{\sqrt{y}} (x^3 - x) dx dy$
- $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2y dy dx$
- $\int_0^6 \int_{y/2}^{5-y/3} 2x dx dy$
- $\int_0^4 \int_1^e \frac{1}{y} dy dx$
- $\int_0^{\pi/2} \int_0^{\sin x} \frac{1}{\sqrt{1-y^2}} dy dx$

35–49 Rework the indicated exercise by reversing the order of integration and verify that the answer does not change. (**Hint:** It is helpful, sometimes even necessary, to sketch the region of integration before reversing the order.)

- Exercise 4
- Exercise 5
- Exercise 6
- Exercise 24
- Exercise 26
- Exercise 28
- Exercise 30
- Exercise 32
- Exercise 34
- Exercise 23
- Exercise 25
- Exercise 27
- Exercise 29
- Exercise 31
- Exercise 33

50–52 Use the symmetry of the graph of f over R to evaluate the double integral. (Do not use repeated integration.)

- $\iint_R x dA$; $R = \{(x, y) \mid -2 \leq x \leq 2, 0 \leq y \leq 4\}$
- $\iint_R \sin y dA$; $R = \{(x, y) \mid -1 \leq x \leq 5, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$
- $\iint_R (x^3 + 1) dA$; $R = \{(x, y) \mid -3 \leq x \leq 3, -1 \leq y \leq 7\}$

53–54 Set up, but do not evaluate, the integral of $f(x, y)$ as an iterated integral over the triangle with the given vertices. Choose the most convenient order of integration. (Answers may vary.)

53. $\iint_R f(x, y) dA$; R is the triangle with vertices $(1, 0)$, $(3, 2)$, and $(4, 0)$.

54. $\iint_R f(x, y) dA$; R is the triangle with vertices $(0, 0)$, $(2, 4)$, and $(4, 3)$.

55. Evaluate $\iint_R 2x dA$ over the region R bounded by $y = \cos x$ and $y = \sin x$, $0 \leq x \leq \pi/4$.

56. Let R be the region between the circle $x^2 + y^2 = 9$ and the ellipse $4x^2 + 9y^2 = 36$. By integrating $f(x, y) = 1$ over R , find the area of the region.

57. Evaluate $\iint_R y dA$ over the region R bounded by $y + x = 2$ and $y = x^2 - 2x$.

58. Evaluate $\iint_R x^2 dA$ over the region R bounded by $y^2 = 5 - x$ and $y^2 = 4x$.

59. Evaluate $\iint_R xy dA$ over the region R bounded by $y = x^3 - 3x$ and $4y = x^3$, $x \geq 0$.

60. Evaluate $\iint_R \sqrt{1-x^2} dA$, where R is the first quadrant of the disk $x^2 + y^2 = 1$.

61. Evaluate $\iint_R y dA$, where R is the circular sector bounded by $y = 0$, $y = \sqrt{3}x$, and $x^2 + y^2 = 4$.

62. Evaluate $\iint_R xe^y dA$, where R is the first quadrant region bounded by $y = \ln x$ and $x = e$.

63–67 The iterated integral represents the (signed) volume of a well-known solid. Use a formula from geometry to evaluate the integral. (Hint: For Exercises 65 and 66, see Exercise 83 of Section 6.1 for the formula for the volume of a generalized cone.)

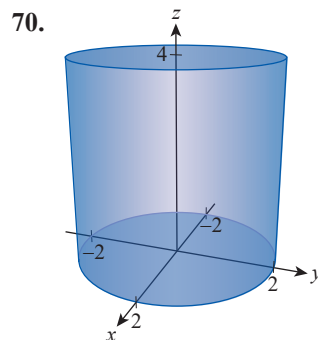
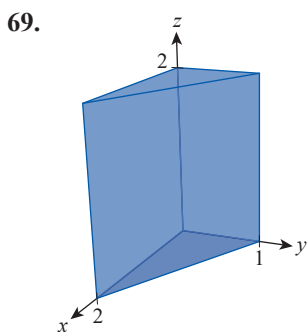
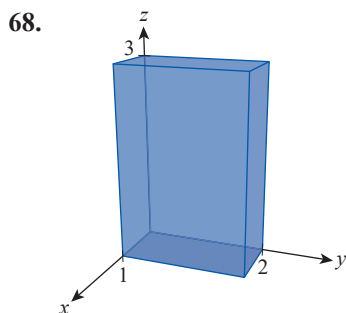
63. $\int_0^4 \int_1^3 2 dy dx$ 64. $\int_{-1}^2 \int_{-3/2}^1 (-3) dx dy$

65. $\int_0^2 \int_0^{4-2x} (4-2x-y) dy dx$

66. $\int_0^1 \int_0^{3-3y} \left(\frac{5}{6}x + \frac{5}{2}y - \frac{5}{2} \right) dx dy$

67. $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2} dx dy$

68–70 Set up and evaluate an appropriate double integral to determine the volume of the solid shown in the figure. Use the order of integration of your choice.



71–74 Evaluate the integral by reversing the order of integration.

71. $\int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi/2}} \cos(x^2) dx dy$ 72. $\int_0^1 \int_x^1 e^{y^2} dy dx$

73. $\int_0^1 \int_y^{\sqrt{y}} \frac{e^x}{x} dx dy$ 74. $\int_0^1 \int_{\sqrt{x}}^1 \sin(y^3) dy dx$

75. Suppose a thin plate has variable density given by a continuous two-variable function $\rho(x, y)$. Interpret the double integral $\iint_R \rho(x, y) dA$.

76. Use the Domination Property to show that if $f(x, y)$ is integrable and nonnegative on the bounded region R , then $\iint_R f(x, y) dA \geq 0$.

77. Use the Domination Property to show that if $f(x, y)$ is bounded on a region R of area A , that is, if $m \leq f(x, y) \leq M$ for some real numbers m, M for all $(x, y) \in R$, then

$$m \cdot A \leq \iint_R f(x, y) dA \leq M \cdot A.$$

78.* Let $F(x, y) = 1$ if at least one (or both) of x and y are irrational and $F(x, y) = 0$ otherwise. Prove that F is not integrable over any bounded region R in \mathbb{R}^2 .

79. Let $R = [a, b] \times [c, d]$, and suppose that the second-order partials of $f(x, y)$ are continuous on an open region containing R .

a. Use the Fundamental Theorem of Calculus to derive a formula for $\iint_R f_{xy}(x, y) dA$.
(Hint: Write the double integral as an iterated integral and use the Fundamental Theorem in two steps.)

b. Letting $f(x, y) = x^3y - 2y^2 + 5xy^4$ and noting that $f_{xy}(x, y) = 20y^3 + 3x^2$ (see Exercise 41 of Section 13.3), use part a. to evaluate

$$\int_{-2}^2 \int_1^3 f_{xy}(x, y) dx dy.$$

80. Let $R = [a, b] \times [c, d]$ and suppose $f(x, y)$ can be decomposed into a product of a function of x and a function of y , that is, $f(x, y) = g(x) \cdot h(y)$. Show that

$$\iint_R f(x, y) = \left[\int_a^b g(x) dx \right] \cdot \left[\int_c^d h(y) dy \right].$$

81–86 Use Exercise 80 to provide a second solution to the given exercise.

81. Exercise 5

82. Exercise 9

83. Exercise 10

84. Exercise 13

85. Exercise 14

86. Exercise 15

87–90. Use the Riemann sum definition to prove the properties of double integrals. (For a refresher on the proofs of the one-variable case, see Section 5.2.)

Concept Check

91–94 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

91. If $f(x)$ is continuous, then

$$\int_c^d \int_a^b f(x) dy dx = (d - c) \int_a^b f(x) dx.$$

92. If R_1 and R_2 are disjoint regions and $f(x, y)$ is continuous, then

$$\left| \iint_{R_1 \cup R_2} f(x, y) dA \right| = \left| \iint_{R_1} f(x, y) dA \right| + \left| \iint_{R_2} f(x, y) dA \right|.$$

93. If $a, b, c > 0$, then $\int_0^c \int_0^{\frac{c-b}{a}y} (c - ax - by) dx dy = \frac{abc}{6}$.

94. $\int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2} dy dx = \frac{4}{3} \pi R^3$