

13.8 Exercises

- Find the absolute extreme values of the function $f(x, y) = 2y^2 - x^2$ on the circle $x^2 + y^2 = 4$.
- Find the absolute extreme values of $f(x, y) = xy$ on the ellipse $\frac{x^2}{4} + \frac{y^2}{8} = 1$.

3–18 Find any extreme values of the function subject to the given constraint.

- $f(x, y) = x^2 + y^2$; constraint: $xy = 2$
- $f(x, y) = x^2 + y^2$; constraint: $2x + y = 5$
- $f(x, y) = x^2 - 3xy + y^2$; constraint: $x^2 + y^2 = 2$
- $f(x, y) = x^2 + 3xy + y^2$; constraint: $y - x = 2$
- $f(x, y) = \frac{x^2 + y^2}{2}$; constraint: $x^2 + 2y^2 = 4$
- $f(x, y) = x^2 + y^2$; constraint: $2x^2 + 3y^2 = 8$
- $f(x, y) = x^2 - y^2$; constraint: $2x + y = 6$
- $f(x, y) = x^2 - y^2$; constraint: $y^2 = 4x$
- $f(x, y) = 2xy$; constraint: $\frac{x^2}{9} + \frac{y^2}{4} = 1$
- $f(x, y) = \sqrt{3x^2 + y^2 - 2}$;
constraint: $x^2 + 2y^2 = 14$
- $f(x, y, z) = 2x - 3y + z$; constraint: $6z = 3x^2 + y^2$
- $f(x, y, z) = x^2 + y^2 + z^2$;
constraint: $2x - 4y + z = 14$
- $f(x, y, z) = x + 2y + 8z$;
constraint: $x^2 + 9y^2 + 16z^2 = 144$
- $f(x, y, z) = xyz$; constraint: $x^2 + y^2 + z^2 = 4$
- $f(x, y, z) = x^2 - 2y^2 + z^2$;
constraint: $2x^2 + y^2 + 3z^2 = 18$
- $f(x, y, z) = x + 2y + 3z$;
constraint: $x^2 + y^2 + z^2 = 1$

19–26 Use Lagrange multipliers to find the coordinates of the point on the given curve that is closest to the point P .

- $y = 2x - 3$; $P(2, 2)$
- $y = 1 - 3x$; $P(-3, 0)$
- $y = x^2$; $P(3, 0)$
- $y = x^2 + x$; $P(2, -2)$

- $y = x^2 + x$; $P(2, 0)$
- $y = 2x - x^2$; $P(1, 0)$
- $x^2 - 2y^3 = 1$; $P(0, 0)$
- $(x^2 + 4)y = 10$; $P(0, 0)$

27–34 Use Lagrange multipliers to find the coordinates of the point on the given surface that is closest to the point P .

- $3x - 2y + \sqrt{3}z + 16 = 0$; $P(0, 0, 0)$
- $z = 3x - 2y + 3$; $P(2, 1, 0)$
- $2z + x - y = 1$; $P(3, -2, -1)$
- $z^2 = x^2 + y^2$; $P\left(1, 0, \frac{1}{2}\right)$
- $z = x^2 + 2y^2$; $P(0, 0, 1)$
- $xyz = 2$; $P(0, 0, 0)$
- $x = yz + 1$; $P(2, 0, 0)$
- $x^2 = yz + 1$; $P(0, 0, 0)$
- Generalize Exercise 27 by showing that the location on the plane $ax + by + cz = d$ closest to the origin is the following point.

$$\left(\frac{ad}{a^2 + b^2 + c^2}, \frac{bd}{a^2 + b^2 + c^2}, \frac{cd}{a^2 + b^2 + c^2} \right)$$

36. Suppose we want to cut a rectangular beam from a log that has a circular cross-section of radius $\sqrt{2}$ feet. Use Lagrange multipliers to find the dimensions of the beam's cross-section if it is to have maximum area. (**Hint:** Place the cross-section in the xy -system with its center at the origin, and maximize the area of the beam's cross-section subject to the constraint $x^2 + y^2 = 2$.)

- Suppose that when manufacturing three different products, a company is able to make a profit of \$4 on each unit of the first product, while the profits on the second and third types of products are \$8 and \$6 per unit, respectively. Let x , y , and z denote thousands of units produced from each product.
 - Find the profit function $P(x, y, z)$.
 - Assuming that the manufacturing process is under the constraint $x^2 + 2y^2 + z^2 \leq 756$, find the maximum profit for the company under these conditions.

38–48 We will revisit some exercises from Section 13.7. Use Lagrange multipliers to provide a second solution for each.

- 38.** Find the dimensions of a rectangular prism with a fixed surface area of 6 square units and maximum volume.
- 39.** Find the dimensions of a rectangular prism with a fixed volume V and minimum surface area.
- 40.** Repeat Exercise 39 for an open box that has no lid.
- 41.** Suppose we want to paint the inside of a rectangular box of volume V and that the paint used on the sides costs \$2 per square unit, the paint for the top is \$3 per square unit, while the sealant used to paint the bottom is \$5 per square unit. What are the dimensions of the box that is the most cost-effective to paint under these conditions?
- 42.** A rectangular box is placed in the three-dimensional coordinate system with one vertex at the origin and the three edges containing it lying along the positive coordinate axes. If the opposite vertex lies in the plane $4x + y + 2z = 9$, what is the greatest possible volume for such a box?
- 43.** Find the dimensions of the rectangular box of maximum volume inscribed in a hemisphere of radius R . (**Hint:** First argue that one of the box's faces should lie in the base plane of the hemisphere.)
- 44.** If the sum of three positive numbers is 300, what is the greatest possible value for their product?
- 45.** Find the minimum distance from the origin to the surface $xyz^2 = 2$.
- 46.** Repeat Exercise 45 for the surface $x^3y^2z = 1$.
- 47.** Repeat Exercise 45 for the surface $z^2 - xy^2 = 3$.
- 48.*** Find the volume of the largest box inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
- 49.** Use Lagrange multipliers to prove that among all triangles inscribed in a circle of radius R , the equilateral triangle has the largest area. (**Hint:** Draw the three radii connecting the center of the circle with the vertices of the triangle, and mark the angles formed by these radii α , β , and γ . Then use the formula for the area of a triangle with sides a , b and included angle θ , $A = \frac{1}{2}ab \sin \theta$.)

50–53 Find the absolute extreme value(s) of the function subject to the given constraints.

50. $f(x, y, z) = x + 2y + 2z$;
constraints: $x^2 + z^2 = 4$ and $2x + y + z = 1$

51. $f(x, y, z) = x^2 + y^2 + z^2$;
constraints: $x + y = 1$ and $2x - y + 3z = 2$

52. $f(x, y, z) = xyz$;
constraints: $x - y + z = 4$ and $x + y + z = 6$

53. $f(x, y, z) = 2x + y - 2z$;
constraints: $x^2 + y^2 = 4$ and $x - y + 2z = 1$

- 54.** Supposing that $x, y, z > 0$, maximize the function $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 1$, and use your result to prove the famous inequality between geometric and arithmetic means for three positive numbers

$$\sqrt[3]{xyz} \leq \frac{x + y + z}{3},$$

with equality occurring precisely when $x = y = z$.

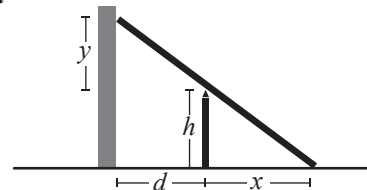
- 55.*** Generalize Exercise 54 for n positive numbers.

- 56.** The graph of the equation $x^2 + xy + y^2 = 4$ is a rotated ellipse. Use Lagrange multipliers to find its points closest to and farthest from the origin.

- 57.** Find the points closest to and farthest from the origin on the intersection of the surfaces $z = 2x^2 + 2y^2$ and $z = 20 - x - y$.

- 58.** Find the highest and lowest points on the curve of intersection of the two surfaces $x^2 + y^2 + z^2 = 9$ and $x - 2y + z = 2$.

- 59.** Suppose a fence of height h is located d units from a wall. Find the minimum length of a ladder that is able to reach the wall over the fence. (**Hint:** Minimize $f(x, y) = (x + d)^2 + (y + h)^2$ subject to the constraint $\frac{h}{x} = \frac{y}{d}$.)



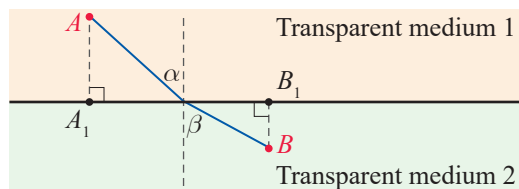
- 60.** Using Lagrange's method, show that the critical point of the function $f(x, y) = x^2y$ subject to the constraint $y - x = 0$ does not yield a local extremum. Explain.

61.* When a light ray travels from a transparent medium into another transparent medium, it “bends” or “refracts” as shown in the figure below. Using the fact that the traveling of light waves is governed by the principle of making the “best time” between A and B , and supposing that the speed of light in the two different media is v_A and v_B , respectively,

use Lagrange’s method to derive the equality

$$\frac{\sin \alpha}{\sin \beta} = \frac{v_A}{v_B}. \quad (\text{This is called Snell’s Law of refraction.})$$

Hint: Minimize the travel time of light subject to the constraint that the distance between A_1 and B_1 is constant.)



13.8 Technology Exercises

62–64 Use a computer algebra system or a programmable calculator to write a program that accepts a three-variable function along with one or two constraints, and uses Lagrange’s method to return the critical points. Use it to check the answers you obtained for the following exercises.

62. Exercise 18

63. Exercise 50

64. Exercise 53