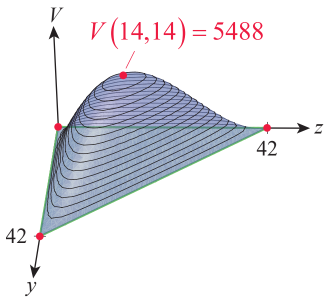


Figure 8

We leave it to the reader (Exercise 32) to show that $V(y,z)=0$ on each edge of the boundary D and also to show that V has a relative (and hence absolute) maximum value at the point $(14,14)$. Thus, the dimensions that maximize volume are a length of 28 inches with a square cross-section of width 14 inches, and the maximum possible volume is 5488 cubic inches. Figure 8 shows a graph of the function V over its triangular region of definition.

13.7 Exercises

1–12 Find any extrema of the given function. Classify critical points without using the Second Partial Derivative Test.

1. $f(x,y)=x^2+4x+y^2-6y+10$

2. $f(x,y)=x^2-12x+y^2+8y$

3. $f(x,y)=10x-x^2-4y-y^2$

4. $f(x,y)=-x^2-y^2-4y+1$

5. $f(x,y)=x^2-8x-y^2-2y+15$

6. $f(x,y)=y^2-x^2-6x$

7. $f(x,y)=y^4-4y^2+x^2+5$

8. $f(x,y)=x^4-2x^2+y^2$

9. $f(x,y)=\dfrac{1}{2+x^4+y^4}$

10. $f(x,y)=\dfrac{1}{x^2+y^2-4y+5}$

11. $f(x,y)=\ln\left(x^2+y^2+2x+2\right)$

12. $f(x,y)=\dfrac{1}{x^2-4x+y^2-2y+6}$

13–30 Use the Second Partial Derivative Test (if necessary) to classify the critical points of the given function. If the test fails, classify the critical point by other means. Identify absolute extrema wherever appropriate.

13. $f(x,y)=x^2+6x+6y^3-8y$

14. $f(x,y)=2x^2-4x+y^2+6y-1$

15. $f(x,y)=y^4-2y^2+4x^2$

16. $f(x,y)=6x^2y-x^2-3y^2$

17. $f(x,y)=x^3+y^3-3x^2-2y^2$

18. $f(x,y)=x^3+2xy^2-y^2$

19. $f(x,y)=3xy-x^3-y^3$

20. $f(x,y)=2xy+\dfrac{4}{x}+\dfrac{1}{y}$

21. $f(x,y)=3-\sqrt[3]{x^2+y^2}$

22. $f(x,y)=\dfrac{x^2+2y^2}{2e^x}$

23. $f(x,y)=2xy-x^4-y^4$

24. $f(x,y)=ye^{x^2-y^2}$

25. $f(x,y)=2xe^{2x-y^2}$

26. $f(x,y)=x\ln(x+y)$

27. $f(x,y)=\left(x^2-y^2\right)e^x$

28. $f(x,y)=e^x-xe^y$

29. $f(x,y)=x\cos y$

30.* $f(x,y)=\sin x+\cos(x+y)$

31. Find and classify all extrema of the function $f(x,y)=\sqrt{x^2+y^2}$.

32. Show that $V(y,z)=0$ on each edge of the boundary of D in Example 6, and demonstrate that V has a relative (and hence absolute) maximum value at the point $(14,14)$.

33–42 Find the absolute extrema of the function on the given region.

33. $f(x,y)=2x+4y-3;$
$D=\left\{(x,y)\middle|-1\le x\le 1, 0\le y\le 2\right\}$

34. $f(x,y)=3x-\dfrac{y}{2}+1;$
$D=\left\{(x,y)\middle|-2\le x\le 3, -1\le y\le 5\right\}$

35. $f(x,y)=2xy-x-y+4$; D: The triangle with vertices $(0,0)$, $(3,0)$, and $(0,3)$

36. $f(x,y)=4xy-2x-y+1$; D: The triangle with vertices $(0,0)$, $(4,0)$, and $(0,2)$

37. $f(x,y)=(x-2y)^2$; D: The triangle with vertices $(0,0)$, $(12,0)$, and $(0,3)$

38. $f(x,y)=1-\sqrt{x^2+y^2}$; $D=\left\{(x,y)\,\middle|\,x^2+y^2\le 1\right\}$

39. $f(x,y)=x^2-2xy+y$; $D=\left\{(x,y)\,\middle|\,-2\le x\le 2,-1\le y\le 1\right\}$

40. $f(x,y)=x^2-2xy+y$; $D=\left\{(x,y)\,\middle|\,0\le x\le 2,x^2\le y\le 4\right\}$

41. $f(x,y)=\dfrac{xy}{(x^2+1)(y^2+1)}$; $D=\left\{(x,y)\,\middle|\,0\le x\le 2,0\le y\le 2\right\}$

42. $f(x,y)=\dfrac{xy}{(x^2+1)(y^2+1)}$; $D=\left\{(x,y)\,\middle|\,0\le x\le 2,0\le y\le \sqrt{4-x^2}\right\}$

43. Find the absolute extrema of the function $g(x,y)=(x^2+2y^2)e^{-(x^2+y^2)}$ on the square $S=\left\{(x,y)\,\middle|\,|x|\le 2,|y|\le 2\right\}$.

44–49 Show that the Second Partial Derivative Test fails for the given function and classify any critical points by other means.

44. $f(x,y)=x^2y^2$

45. $g(x,y)=\dfrac{1}{x^2+y^2}$

46. $h(x,y)=(x-1)^3+(y+2)^3$

47. $F(u,v)=(u+1)^{2/3}+(v-1)^{2/3}$

48. $R(s,t)=s^3+t^3-3t^2-2$

49. $k(x,y)=x^3-2x^2y$

50. Demonstrate that even though the function $f(x,y)=4+2x-x^2$ has infinitely many critical points, the Second Partial Derivative Test fails to classify any of them. Are those points extrema, and if so, what kind?

51. Determine m and b such that the sum of the squares of vertical distances from the line $y=mx+b$ to the points $(0,3)$, $(1,1)$, and $(4,8)$ is minimal.

52. Repeat Exercise 51 for the points $(1,0)$, $(2,5)$, and $(6,9)$.

53. Find a line that minimizes the sum of squares of the horizontal distances between the points in Exercise 51 and the line.

54. Find a line that minimizes the sum of squares of the horizontal distances between the points in Exercise 52 and the line.

55. By generalizing Exercise 51 to n points, this exercise will prove the formulas used in the least-squares method of curve fitting (see Section 1.5). Given n data points (x_1,y_1), (x_2,y_2), ..., (x_n,y_n) in the plane, let $y=mx+b$ be the line that minimizes the sum of the squares of the vertical distances from the line to the points.

a. Using the notation $S(m,b)=\sum_{i=1}^{n}\left[y_i-f(x_i)\right]^2$, show that in order for $S(m,b)$ to be minimal, m and b have to satisfy

$$\left(\sum_{i=1}^{n}x_i\right)m+nb=\sum_{i=1}^{n}y_i$$

and

$$\left(\sum_{i=1}^{n}x_i^2\right)m+\left(\sum_{i=1}^{n}x_i\right)b=\sum_{i=1}^{n}x_iy_i.$$

b. Solve the above system to derive the formulas introduced in Section 1.5 for m and b.

56. Find the minimum distance from the origin to the surface $xyz^2=2$.

57. Repeat Exercise 56 for the surface $x^3y^2z=1$.

58. Repeat Exercise 56 for the surface $z^2-xy^2=3$.

59. Use the methods of this section to find the minimum distance between the point $(-3,-4,1)$ and the plane $4x+y-2z=3$.

60. Find the dimensions of a rectangular prism with a fixed surface area of 6 square units and maximum volume.

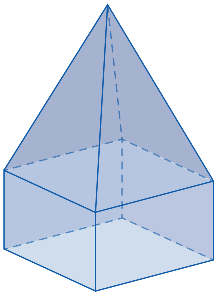
61. Find the dimensions of a rectangular prism with a fixed volume V and minimum surface area.

62. Repeat Exercise 61 for an open box that has no lid.

63. Suppose we want to paint the inside of a rectangular box of volume V and that the paint used on the sides costs \$2 per square unit, the paint for the top is \$3 per square unit, while the sealant used to paint the bottom is \$5 per square unit. What are the dimensions of the box that is the most cost-effective to paint under these conditions?

64. A rectangular box is placed in the three-dimensional coordinate system with one vertex at the origin and the three edges containing it lying along the positive coordinate axes. If the vertex opposite the origin lies in the plane $4x + y + 2z = 9$, what is the greatest possible volume for such a box?

65.* A right rectangular pyramid with a square base is sitting atop a right rectangular prism with a congruent base to form the solid seen in the given figure. Find the side length of the base, and the respective heights of the pyramid and the prism that minimize the lateral surface area of the solid, if its volume is $\frac{100}{3}$ cubic units.



66. Find the dimensions of the rectangular box of maximum volume inscribed in a hemisphere of radius R. (**Hint:** First argue that one of the box's faces should lie in the base plane of the hemisphere.)

67.* Find the volume of the largest box inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

68. If the sum of three positive numbers is 300, what is the greatest possible value for their product?

69. Again, as in Exercise 68, assume the sum of three positive numbers is 300. Find the minimum value for the sum of their squares.

70. A 30 in. piece of wire is cut into three pieces which are then bent into squares. If possible, determine how to **a.** minimize, and **b.** maximize the sum of the areas of the three squares.

71. Suppose $a, b, c > 0$ are given and consider the tetrahedron formed by the three coordinate planes and an arbitrary plane containing the point (a,b,c). Find the minimum volume for such a tetrahedron.

72.* Find the coordinates of the point Q so that the sum of the squares of the distances between Q and the given points $(x_1, y_1), \ldots, (x_n, y_n)$ is minimal.

73. Show that the function $f(x,y) = 3x^4 - 8x^2y + 4y^2$ has a relative minimum along every line $y = mx$ through the origin, but $(0,0)$ is *not* a relative minimum for f. (**Hint:** Examine the behavior of f along the parabola $y = x^2$.)

Concept Check

74–78 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

74. If $f(x,y)$ has a relative minimum at (a,b), then $f_x(a,b) = f_y(a,b) = 0.$

75. If $f_x(a,b) = f_y(a,b) = 0,$ then $f(x,y)$ has a relative extremum at (a,b).

76. If $f(x,y)$ has a relative extremum at an interior point (a,b) of its domain, and if f_x and f_y both exist at (a,b), then $f_x(a,b) = 0 = f_y(a,b)$.

77. If $f(x,y)$ has exactly two relative maxima, then it must have a relative minimum also.

78. If $f(x,y)$ has an extremum at (a,b) along every straight line $y = mx$, then (a,b) is an extremum for f.

13.7 Technology Exercises

79. With the help of a graphing utility, create an example of a two-variable function that has two maxima, but no relative minima. Sketch the graph of your example. (To start off, you may want to review, for example, Exercise 7, 15, or 23 of this section. Answers will vary.)

80–85 Use a graphing utility to graph the function of the indicated exercise and graphically reinforce your conclusions made in the referenced exercise.

80. Exercise 7

82. Exercise 21

84. Exercise 30

81. Exercise 15

83. Exercise 29

85. Exercise 43