

### Example 7 Linearizing a Function of Four Variables

Find the linear approximation of the following function at the given point.

$$f(w, x, y, z) = xy^2 - \frac{3w}{z} + \sin(x^3 yz); \quad (-2, 1, 0, 3)$$

#### Solution

The linearization of  $f$  at  $(-2, 1, 0, 3)$  is given by the formula

$$L(w, x, y, z) = f(-2, 1, 0, 3) + f_w(-2, 1, 0, 3)(w+2) + f_x(-2, 1, 0, 3)(x-1) \\ + f_y(-2, 1, 0, 3)(y-0) + f_z(-2, 1, 0, 3)(z-3),$$

so we calculate the following values of the function and its four partial derivatives.

$$f(-2, 1, 0, 3) = (1)(0)^2 - \frac{3(-2)}{3} + \sin((1)^3(0)(3)) = 2$$

$$f_w(-2, 1, 0, 3) = \left[ -\frac{3}{z} \right]_{(-2, 1, 0, 3)} = -1$$

$$f_x(-2, 1, 0, 3) = \left[ y^2 + (3x^2 yz) \cos(x^3 yz) \right]_{(-2, 1, 0, 3)} = 0$$

$$f_y(-2, 1, 0, 3) = \left[ 2xy + (x^3 z) \cos(x^3 yz) \right]_{(-2, 1, 0, 3)} = 3$$

$$f_z(-2, 1, 0, 3) = \left[ \frac{3w}{z^2} + (x^3 y) \cos(x^3 yz) \right]_{(-2, 1, 0, 3)} = -\frac{2}{3}$$

Hence,

$$L(w, x, y, z) = 2 - (w+2) + 3y - \frac{2}{3}(z-3) \\ = -w + 3y - \frac{2}{3}z + 2.$$

## 13.6 Exercises

**1–8** Determine the tangent plane and normal line at the indicated point of the given surface.

- $z - x^2 + 2y^2 = 0; \quad (1, 1, -1)$
- $z - xy + y^3 x^2 = 0; \quad (2, -1, 2)$
- $z^2 + (3x+1)(y-x^2) = 0; \quad (1, -3, 4)$
- $z^2 - 3x^2 - y^2 = 15; \quad (2, -3, 6)$
- $(xy+2)^2 + 3(y+1)^2 - \sqrt{z} = 5; \quad (2, -2, 4)$
- $xyz = 18; \quad (-3, 2, -3)$
- $x^2 z - x^2 y^2 = 3; \quad (1, -3, 12)$
- $z = \sin(2xy) - 5; \quad \left( \frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}, -4 \right)$

**9–16** Determine the tangent plane at the indicated point of the given surface.

- $z = xy^2 - x^3 y; \quad (1, 2, 2)$
- $z = \tan(xy) + 2; \quad \left( 1, \frac{\pi}{4}, 3 \right)$
- $z = \frac{y}{\sqrt{x}}; \quad (9, 3, 1)$
- $z = \ln(x^2 + 2y^2); \quad (1, 1, \ln 3)$
- $z = \frac{1}{x^2} + y^2; \quad (1, 3, 10)$
- $z = 2 \sin x \cos y; \quad \left( \frac{\pi}{4}, \frac{\pi}{4}, 1 \right)$
- $z = \frac{1}{\sqrt{x^2 + y^2}}; \quad \left( 3, 4, \frac{1}{5} \right)$
- $z = e^{y/x}; \quad (3, 6, e^2)$

17. Given a differentiable function  $f(x, y)$ , use your knowledge of partial derivatives to show that the vectors  $\mathbf{u} = \langle 1, 0, f_x(x_0, y_0) \rangle$  and  $\mathbf{v} = \langle 0, 1, f_y(x_0, y_0) \rangle$  are direction vectors for the plane tangent to the surface  $z = f(x, y)$  at the point  $(x_0, y_0, f(x_0, y_0))$ . Then use the cross product to find a normal vector  $\mathbf{n}$  and arrive at the following equation of the tangent plane that we derived immediately preceding Example 2.

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

18. Assuming that  $F(x, y, z)$  is differentiable and  $F(x, y, z) = 0$  defines a function  $z = f(x, y)$  implicitly, use the formulas  $f_x = -F_x/F_z$  and  $f_y = -F_y/F_z$  to provide yet another derivation of the equation of the tangent plane described in Exercise 17.

**19–22** Find parametric equations for the line tangent to the intersection of the two surfaces at the indicated point.

19.  $z = \frac{x^2 + y^2}{5}$ ;  $z = 8 - 3x^2 - y^2$ ;  $(1, 2, 1)$
20.  $3x^2 + y^2 - z^2 = 2$ ;  $x^2 + y^2 + z^2 = 14$ ;  $(2, 1, 3)$
21.  $x^2 + 5y^2 + 2z^2 = 11$ ;  $xyz = 2$ ;  $(-2, 1, -1)$
22.  $4x^3 + y^3 + 5x^2y - xy = z^2$ ;  $4x^2 - y^2 - z = 0$ ;  $(1, 1, 3)$
23. Find a parametrization  $\mathbf{r}(t)$  of the curve  $C$  of Example 3; then determine  $\mathbf{r}'$  at the point  $(1, 1, 1)$ , and use it as a direction vector to obtain parametric equations for the tangent line to  $C$  at  $(1, 1, 1)$ .
24. Show that, in general, the tangent to curve  $C$  at the point  $(x, y, z)$  of the intersection of the surfaces in Example 3 points in the direction of  $\langle -10y, 6x, -4xy \rangle$ . (**Hint:** Determine the cross product of  $\nabla F$  and  $\nabla G$ .)
25. Find all points on the surface  $8x^2 + 4y^2 + z^2 = 4$  where the tangent plane is normal to the vector  $\langle 4, 2, 1 \rangle$ . (Notice that this surface is an ellipsoid. How many such points do you expect?)
26. Repeat Exercise 25 for the hyperboloid  $x^2 + 3y^2 - 2z^2 = 6$  and the vector  $\langle 1, 3, 2 \rangle$ .

**27–34** Find **a.** the differential  $df$  at an arbitrary point  $(x, y)$  and **b.** the linearization of  $f$  at the indicated point.

27.  $f(x, y) = x^3 + y - (y + 2)^2$ ;  $(1, 3)$
28.  $f(x, y) = xy^3 - x^2y^2 + x^3y$ ;  $(-2, 1)$

29.  $f(x, y) = \sqrt{x^2 + y^2}$ ;  $(\sqrt{3}, -1)$
30.  $f(x, y) = x \cos y + y \cos x$ ;  $(\pi, 0)$
31.  $f(x, y) = \frac{e^{x^2 - y^2}}{2}$ ;  $(1, 1)$
32.  $f(x, y) = \arctan \frac{x}{y}$ ;  $(\sqrt{3}, 1)$
33.  $f(x, y) = 2(\sqrt{x} - \sqrt{y})$ ;  $(1, 4)$
34.  $f(x, y) = e^{3x} \cos 2y$ ;  $(0, \pi/2)$

**35–39** Find the linear approximation of the function at the indicated point.

35.  $f(w, x, y, z) = 2x^2(wy - z^3) + \frac{4wx}{yz}$ ;  $(3, -1, 1, 2)$
36.  $f(x, y, z) = \frac{xyz}{x^2 + y^2 + z^2}$ ;  $(2, 1, -1)$
37.  $f(w, x, y, z) = e^{\sqrt{w^2 + x^2 + y^2 + z^2}}$ ;  $(-2, 2, 0, -1)$
38.  $f(w, x, y, z) = \arctan(wxyz)$ ;  $(-1, 1, 1, -1)$
39.  $f(x, y, z) = e^x \sqrt{\ln(y - z)}$ ;  $(1, e, 0)$

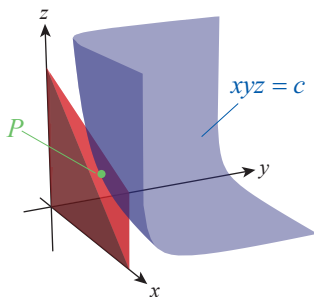
40. Using the linear approximation of the function  $f(x, y) = \sqrt{x^2 - y^2}$  at the point  $(5, 4)$ , find an approximation for the number  $\sqrt{(4.9)^2 - (4.1)^2}$ . Compare your approximation with the value returned by your calculator.

**41–46** Find an appropriate function and mimic the process from Exercise 40 to approximate the given number. Compare your approximation with the value returned by your calculator. (In Exercises 45–46, generalize to three variables.)

41.  $\frac{(1.98)^3}{(3.01)^2 - 1}$
42.  $\sqrt{\frac{16.5}{24.2}}$
43.  $\frac{1}{\sqrt{(3.8)^2 + (3.1)^2}}$
44.  $\frac{6.16}{2.9}$
45.  $\sqrt{(8.98)(3.01)(2.9)}$
46.  $\frac{\ln(2.8)}{(2.15)(4.95)}$

47. Suppose the legs of a right triangle are known to within 3% relative error and within 4% relative error, respectively. Find an upper bound on the relative error with which the triangle's area is known.

48. Suppose the length and width of the base of a rectangular pyramid were measured within a relative error of 1%, while its height is known to within 3% relative error. What is the greatest possible relative error this causes in the volume of the pyramid?
49. Repeat Exercise 48 for a circular cone if the radius of the base is measured within a relative error of 1%, while its height is known to within 3% relative error.
50. The electrical power used by a resistor is given by the equation  $P = V^2/R$ , where  $R$  is its resistance and  $V$  is the voltage. Suppose  $R$  and  $V$  are known to within 2% and 1% relative error, respectively. Find an upper bound on the relative error with which the power is known.
51. As we have seen earlier (see, for example, Exercise 17 in Section 3.7), the speed of impact of a body falling from a height  $h$  in the absence of air resistance is  $v = \sqrt{2hg}$ . Find the maximum relative error if one uses  $g \approx 10$  m/s (instead of its “true value” of 9.81 m/s), and if  $h$  is only known to within 3% of its true value.
52. Consider the first-octant tetrahedron formed by the coordinate planes and the plane tangent to the surface  $xyz = c$  at a fixed point  $P$  on the surface. Find the volume of this tetrahedron and show that its value is independent of the choice of  $P$ .



53. Show that every plane tangent to the elliptic cone  $x^2 + y^2 - z^2 = 0$  passes through the origin.
54. Suppose you are working on a mathematical model that involves positive numbers less than 1000 and that your calculator rounds all your data to two decimal places. Use linear approximation to estimate the maximum error this might cause when multiplying three numbers.
55. Consider the two moles of ideal gas in the 50-liter container in Exercise 70 of Section 13.4. Use linear approximation to estimate the change in pressure if the temperature increases from 323 K to 328 K and the volume increases to 51 liters.
- 56.\* Suppose four resistors are connected in parallel, with the following resistances:  $R_1 = 12$  ohms ( $\Omega$ ),  $R_2 = 16$   $\Omega$ ,  $R_3 = 9$   $\Omega$ , and  $R_4 = 6$   $\Omega$ . If the actual resistances might differ by up to 2% from the above specifications, use linear approximation to estimate the maximum relative error in the calculated value of the net resistance. (See Exercise 101 of Section 13.3.)
57. Recall that the standard equation of the ellipsoid centered at the origin is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ . Show that the tangent plane to its surface at the point  $(x_0, y_0, z_0)$  has equation  $\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = 1$ .
58. Find the equation of the tangent plane to the hyperboloid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$  at  $(x_0, y_0, z_0)$ . (See Exercise 57.)

## Concept Check

**59–62** Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

59. If both  $f_x(a, b)$  and  $f_y(a, b)$  exist, then a tangent plane to the graph of  $f(x, y)$  exists at the point  $(a, b)$ .
60. If a tangent plane to the graph of  $f(x, y)$  exists at the point  $(a, b)$ , then every directional derivative of  $f$  exists at  $(a, b)$ .
61. If  $f(x, y)$  is differentiable at  $(a, b)$ , then  $\nabla f(a, b)$  is contained in the tangent plane and points in the direction of greatest increase of  $f$ .
62. If  $(x_0, y_0, z_0)$  is a point on the level surface  $F(x, y, z) = k$  of a differentiable function  $F(x, y, z)$ , then  $\nabla F(x_0, y_0, z_0)$  is normal to the plane tangent to that level surface at  $(x_0, y_0, z_0)$ .