Solution

We begin with the gradient.

$$\nabla T(x,y,z) = -100\frac{\langle 4x, 2y, z\rangle}{\left(1+2x^2+y^2+0.5z^2\right)^2}$$

The direction of the greatest decrease in rate of change of T from the point $(1,3,2)$ is then

$$-\nabla T(1,3,2) = 100\frac{\langle 4,6,2\rangle}{(1+2+9+2)^2} = \frac{100}{14^2}\langle 4,6,2\rangle \approx \langle 2.04, 3.06, 1.02\rangle,$$

and the rate of decrease in that direction is approximately $-3.82\ ^\circ\text{C/m}$.

13.5 Exercises

1–6 Find the gradient of the function at the indicated point.

1. $f(x,y) = 5x^3 - 2y^2;\quad (1,2)$
2. $g(x,y) = 2xy + \frac{\sqrt{y}}{x};\quad (1,4)$
3. $h(x,y) = e^{x^2+y^2};\quad (-1,1)$
4. $k(x,y) = \frac{y}{\sqrt{x^2+y^2}};\quad (-4,3)$
5. $F(x,y,z) = x\sqrt{\frac{y^3}{z}};\quad (-2,3,3)$
6. $G(x,y,z) = \sqrt{2}x\sin\left(\pi y + z^2\right);\quad \left(3,\frac{1}{4},0\right)$

7–12 Use the definition to find the directional derivative of *f* at the given point in the direction of the given vector. Then compute the directional derivative with the help of the gradient vector to check your answer.

7. $f(x,y) = 2x - y;\quad (0,0);\quad \mathbf{v} = \langle 3,4\rangle$
8. $f(x,y) = x^2y;\quad (1,-1);\quad \mathbf{v} = \langle 1,2\rangle$
9. $f(x,y) = \frac{x}{y};\quad (1,2);\quad \mathbf{v} = \langle 2,2\rangle$
10. $f(x,y) = x\sqrt{y};\quad (4,1);\quad \mathbf{v} = \langle 1,\sqrt{3}\rangle$
11. $f(x,y,z) = 2xy - z^2;\quad (1,1,1);\quad \mathbf{v} = \langle 1,2,3\rangle$
12. $f(x,y,z) = xyz;\quad (5,0,-2);\quad \mathbf{v} = \langle 2,4,\sqrt{5}\rangle$
13. Try to evaluate the limit in Example 1 by using the following parametrizations.

    a. $\langle t,-t\rangle$ **b.** $\langle 2t,-2t\rangle$

    What do you find?

14–29 Find the derivative of the function at the given point in the indicated direction. (Note that sometimes the direction is conveniently specified in terms of a direction angle, the one determined by a direction vector and the positive *x*-axis. In this case, the corresponding unit vector is $\mathbf{u} = \langle \cos\alpha, \sin\alpha\rangle$.)

14. $f(x,y) = x^2 - 4y^2;\quad (1,2);\quad \mathbf{v} = \langle 3,4\rangle$
15. $f(x,y) = 3x^2 - 2xy + y^2;\quad (0,1);\quad \mathbf{v} = \langle 1,1\rangle$
16. $f(x,y) = 2x^3y^2;\quad (1,2);\quad \mathbf{v} = \langle -2,1\rangle$
17. $f(x,y) = \frac{x^3}{3} + x^2y - 3xy^2 + y^3;\ (-1,-1);$ $\mathbf{v} = \langle -4,-3\rangle$
18. $f(x,y) = ye^x;\quad (1,0);$ in the direction toward $(-1,4)$
19. $f(x,y) = 4e^x\cos x;\quad (0,0);\quad \alpha = 60^\circ$
20. $f(x,y) = \arcsin(xy);\quad (0,1);$ in the direction toward $(3,4)$
21. $f(x,y) = 2\cos x\sin y;\quad \left(-\frac{\pi}{6},\frac{5\pi}{6}\right);\quad \mathbf{v} = \langle 5,12\rangle$
22. $f(x,y) = \frac{2x}{\sqrt{x^2+y^2}};\quad (0,1);\quad \mathbf{v} = \langle 0.7,2.4\rangle$
23. $f(x,y) = \ln\left(x^2+y^2\right);\quad (1,0);\quad \alpha = 30^\circ$
24. $f(x,y,z) = xy - yz + xz;\quad (1,-3,-2);\quad \mathbf{v} = \langle 2,1,-2\rangle$
25. $f(x,y,z) = e^{-\left(x^2+y^2+z^2\right)};\quad (2,1,1);\quad \mathbf{v} = \langle -6,2,3\rangle$
26. $f(x,y,z) = \sqrt{x^2+y^2+z^2};\quad (1,1,1);\quad \mathbf{v} = \langle 1,1,1\rangle$
27. $f(x,y,z) = x\arctan\frac{y}{z};\quad (1,1,1);\quad \mathbf{v} = \langle 2,-2,-2\rangle$

28. $f(x,y,z)=\ln(x+y+z)$; $(e,1,-1)$; in the direction toward $(e-1,1,1)$

29. $f(x,y,z)=ze^{xy}$; $(2,0,1)$; in the direction toward $(-1,4,1)$

30–33 Find the direction and value of the greatest rate of increase for the function at the given point.

30. $f(x,y)=2x^2-5xy+y^2$; $(2,1)$

31. $f(x,y)=x^2e^{2xy}$; $(1,0)$

32. $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$; $(12,5)$

33. $f(x,y,z)=\dfrac{x-y}{z+2}$; $(4,2,0)$

34–37 Find the direction and value of the greatest rate of decrease for the function at the given point.

34. $f(x,y)=\dfrac{x^2+y^2}{x^2-y^2}$; $(-1,2)$

35. $f(x,y)=\cos(\pi xy)$; $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$

36. $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$; $(2,1,-2)$

37. $f(x,y,z)=xe^{-yz}$; $(-1,3,0)$

38–39 Find the direction of no change for the function at the given point.

38. $f(x,y)=x^2y+2x^3-3y^2$; $(-1,1)$

39. $f(x,y)=\dfrac{x-y}{x^2+y^2}$; $(1,-1)$

40. If possible, find a direction angle θ for which the rate of change at the point $(1,-1)$ of the function in Exercise 39 is **a.** 0, **b.** 1.

41–48 Find an equation for the line tangent to the graph of the given equation at the indicated point.

41. $x^2-xy+y^2=7$; $(1,3)$

42. $(6-x)y^2=2x^3$; $(2,2)$

43. $(x^2+4)y=10$; $(-1,2)$

44. $x^{2/3}+y^{2/3}=13$; $(8,27)$

45. $(2x^2+y^2)^2-9x^2y=0$; $(1,1)$

46. $x^2+y^2=(x^2+y^2-5x)^2$; $(4,3)$

47. $(x^2+y^2)^2=4xy$; $(1,1)$

48. $\dfrac{y}{x^2+y^2}=\dfrac{3}{4}+x^2$; $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$

49–52 Generalize Example 4 to three variables to obtain the equation of the tangent plane to the surface at the given point.

49. $x^2+3y^2+4z^2=11$; $(2,1,-1)$

50. $y^3z^2+5xz^2+2xy=32$; $(2,-1,2)$

51. $x^2-2y^2=z$; $(3,2,1)$

52. $3x-2y^2=3z^2$; $(7,3,1)$

53. Prove the linearity of the gradient, that is, the Sum/Difference Law and the Constant Multiple Law: If f and g are differentiable functions and k is a real number, then $\nabla(f\pm g)=\nabla f\pm\nabla g$, and $\nabla(kf)=k\nabla f$.

54. Prove the Product Law for gradients, that is, if f and g are differentiable functions, then $\nabla(fg)=f\nabla g+g\nabla f$.

55. Prove the Quotient Law for gradients, that is, if f and g are differentiable functions and $g\neq 0$, then $\nabla\left(\dfrac{f}{g}\right)=\dfrac{g\nabla f-f\nabla g}{g^2}$.

56–59 Use the properties of the gradient to determine ∇f.

56. $f(x,y)=\dfrac{2xy^2+yx^3}{x^2+2xy}$

57. $f(x,y)=\dfrac{2x^3-\sqrt{y}}{x(x+y)}$

58. $f(x,y)=\dfrac{xy}{x^3-y^3}$

59. $f(x,y,z)=\dfrac{xyz}{x^2+y^2+z^2}$

60–65 Find a function with the given gradient. If it is not possible, explain why. (Answers will vary.)

60. $\nabla f=\langle 1,2\rangle$

61. $\nabla f=\langle 2y,2x\rangle$

62. $\nabla f=\langle 3y,x\rangle$

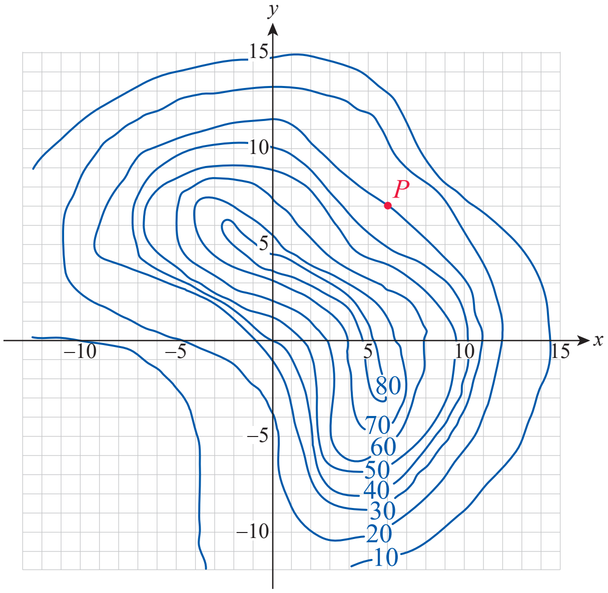
63. $\nabla f=\langle y,2x^2\rangle$

64. $\nabla f=\langle 2y,2x,2z\rangle$

65. $\nabla f=\langle 6xyz^2,3x^2z^2,6x^2yz\rangle$

66. Prove that the rate of change of the function $f(x,y)=\sqrt{x^2+y^2}$ is greatest along rays emanating from the origin.

67. Use the contour map below to estimate the directional derivative $D_{\mathbf{u}}f$ at the point $P(6,7)$ in the direction of **a.** $\mathbf{u}=\langle 1,2\rangle$, **b.** $\mathbf{u}=\langle -1,-1\rangle$. Then draw a possible path of steepest ascent starting at *P*. (Answers will vary.)



68. Sarah is standing at an intersection on a mountain trail where, according to the trail markings, her route continues to the southeast. Her current position can be modeled by the point $P(400,200,3560)$ on the graph of $f(x,y)=4000-0.002x^2-0.003y^2$ (units in feet).

a. What will be the angle of elevation (or depression) of her route immediately after leaving the intersection?

b. What is the direction and angle of steepest ascent from her current position? (Assume that the northern direction coincides with the positive *y*-axis.)

69. Suppose that the temperature of a metal plate is given by

$$T(x,y)=\frac{150}{\sqrt{x^2+y^2+1}}.$$

Find the rate of change of temperature at the point $(8,4)$ in the direction toward the point $(7,2)$.

70. Suppose that the temperature around the origin in three-dimensional space is given by $T(x,y,z)=300e^{-(x^2+2y^2+3z^2)}$.

a. Find the rate of change of temperature at the point $(2,1,0)$ in the direction toward the point $(4,0,2)$.

b. Find the direction at $(2,1,0)$ in which the rate of decrease is greatest.

c. Find the rate of greatest decrease at the point $(2,1,0)$.

71.* Consider a path on the contour map of a differentiable two-variable function $f(x,y)$ that follows the gradient at each point. Such is a possible path of a heat-seeking object, if *f* were a temperature function, or a path of steepest ascent on a geographical map (see Exercise 67). If such a path is parametrized as $\langle x(t),y(t)\rangle$, prove that

$$\frac{y'(t)}{x'(t)}=\frac{f_y}{f_x}.$$

72. Use Exercise 71 to find the path of a heat-seeking object if it starts at the point $(5,25)$ on a plane whose temperature is given by the function $T(x,y)=500-x^2-3y^2$. (**Hint:** Notice that this is a separable initial value problem. See Section 8.1.)

73. Use Exercise 71 to find the equation in the *xy*-plane of the steepest path from the point $(400,200)$ (this is the projection of the spot where Sarah is standing) for the function in Exercise 68. (See the hint given in Exercise 72.)

74.* Consider the piecewise–defined function of Exercise 114 of Section 13.3.

$$F(x,y)=\begin{cases}\dfrac{xy^2}{2(x^2+y^2)} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Prove that all directional derivatives of *F* exist at $(0,0)$, but *F* is not differentiable at $(0,0)$. (**Hint:** To prove the existence of the directional derivatives, use the definition.)

75.* Suppose $f(x,y)$ is defined and differentiable on an open region *R*, and $\nabla f(x,y)=\mathbf{0}$. Prove that $f(x,y)$ is constant on *R*.