$$z_x = -\frac{F_x}{F_z} = -\frac{2xy - z^3 - yz}{-3xz^2 - xy} = \frac{2xy - z^3 - yz}{3xz^2 + xy}$$

$$z_y = -\frac{F_y}{F_z} = -\frac{x^2 - xz}{-3xz^2 - xy} = \frac{x^2 - xz}{3xz^2 + xy}$$

# 13.4 Exercises

**1.** Obtain the result of Example 1 by first expressing $y$ explicitly as a function of $x$.

**2–5** Use a tree diagram to apply the Chain Rule to express the indicated derivative of the given function.

**2.** $\frac{df}{dt}$; $f = f(x(t), y(t))$

**3.** $\frac{dg}{dt}$; $g = g(x(t), y(t), z(t))$

**4.** $\frac{\partial h}{\partial u}$; $h = h(x(u,v), y(u,v), z(u,v))$

**5.** $\frac{\partial k}{\partial z}$; $k = k(u(x,y,z), v(x,y,z))$

**6–17** Determine $dy/dx$, given $y$ as a function of $u(x)$ and $v(x)$. In Exercises 6–13, check your answer by expressing $y$ explicitly as a function of $x$ and differentiating. In Exercises 14–17, generalize to three intermediate variables.

**6.** $y = u^2 v$; $u = 3x + 1$, $v = x^4$

**7.** $y = uv^2 - \cos u$; $u = 2x^2$, $v = \sqrt{x}$

**8.** $y = u^3 v - \sin u$; $u = 2x$, $v = x^2$

**9.** $y = \ln \frac{u}{\sqrt{v}}$; $u = \sin x$, $v = \cos^2 x$

**10.** $y = \sqrt{u^2 + v^2}$; $u = e^x \cos x$, $v = e^x \sin x$

**11.** $y = \frac{1}{u} + \frac{1}{v}$; $u = \csc^2 x$, $v = \cot x$

**12.** $y = u \arctan v$; $u = e^x$, $v = \tan x$

**13.** $y = \arcsin \frac{u}{v}$; $u = 2x$, $v = x^3$

**14.** $y = uv + uw + vw$; $u = 2x$, $v = x + 2$, $w = x^2 + 2$

**15.** $y = u \sin^2(vw)$; $u = 2x$, $v = x^2$, $w = \frac{1}{x}$

**16.** $y = uvw$; $u = e^x$, $v = 3x$, $w = x^3$

**17.** $y = uve^w$; $u = 5x - 2$, $v = \sin x$, $w = \ln x$

**18.** Determine the value of $f'(7\pi/6)$ for the function in Example 2.

**19–21** After determining the rate of change with respect to $t$ of the function $f(x, y)$ along the indicated parametric curve, find $f'(t)$ at the given point.

**19.** $f(x, y) = x^2 y$ along the curve $x = 10\cos t$, $y = t$, at $t = \pi/2$

**20.** $f(x, y) = (x + y)^2$ along the curve $x = t\cos t$, $y = t\sin t$, at $t = \pi$

**21.** $f(x, y) = xe^{-y}$ along the curve $x = 4 - 2t^2$, $y = t^3 - 9t$, at $t = 0$

**22.** Find the rate of change of the function $f(x, y, z) = x^2 y + \sin z$ along the helix $x = \cos 2t$, $y = \sin 2t$, $z = t/2$. Express your answer in terms of the variable $t$.

**23–33** Use the Chain Rule to determine the partial derivatives $z_x$ and $z_y$. (Answers may be left in terms of the intermediate and independent variables.)

**23.** $z = u^2 + v^2$; $u = x + 2y$, $v = y - x$

**24.** $z = u \sin v$; $u = x^2 + y$, $v = x - y^2$

**25.** $z = v^3 - 2u^2 v$; $u = \cos y$, $v = \sin x$

**26.** $z = v^2 - 2uv$; $u = x \sin y$, $v = y \cos x$

**27.** $z = 2v^4 - 3u\sqrt{v}$; $u = e^x$, $v = e^y$

**28.** $z = \cos(u + v^2)$; $u = y - \frac{x^2}{2}$, $v = 3x - 2y$

**29.** $z = u^2 v - uw^3$; $u = x + y$, $v = x^2 - y$, $w = 2xy$

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**30.** $z = u(v+w)^2 - v^2;\quad u = 3x,\quad v = 5y^2,\quad w = x - 2y$

**31.** $z = v^2 e^{2u+3w};\quad u = x - y,\quad v = 2y - x,\quad w = 3xy$

**32.** $z = \ln(u^2 + v^2 + w^2);\quad u = 2x - y,\quad v = xy,\quad w = e^y$

**33.** $z = v\sin(uw^2);\quad u = x + y^3,\quad v = 3y - x,\quad w = x^2 y$

**34.** Find $w_\rho$, $w_\theta$, and $w_\phi$ if $w = 2x + y^2 z$ and $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\theta$. (Answers may be left in terms of the intermediate and independent variables.)

**35.** Prove that for $z = f(x(u,v), y(u,v))$ of Example 5,

$$z_{vv} = 2z_x + 4v^2 z_{xx} + 4uvz_{xy} + u^2 z_{yy}.$$

**36.** Assume (as in Example 5) that $z = f(x,y)$ has continuous second-order partial derivatives and that $x = u^2 v$ and $y = u + v^2$. Find $z_{uu}$ and $z_{vv}$.

**37.** Suppose $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Prove the following.

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

(**Hint:** Start by determining $f_r$ and $f_\theta$.)

**38.** Suppose $z = f(x,y)$ is as in Exercise 37. Prove that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\cdot\frac{\partial f}{\partial r}.$$

(This is called the *Laplacian* of $f$. Can you see why?)

**39–47** Find $dy/dx$, where $y$ is given implicitly by the given equation.

**39.** $x^2 - xy + y^2 = \frac{1}{4}$

**40.** $(4-x)y^2 = 2x^3$

**41.** $(x^2+4)y = 8$

**42.** $\frac{y^2}{4}(8-x) = x^3$

**43.** $x^{2/3} + y^{2/3} = 8$

**44.** $(2x^2 + y^2)^2 - 4x^2 y = 0$

**45.** $x^2 + y^2 = (x^2 + y^2 - 3x)^2$

**46.** $(x^2 + y^2)^2 = 9xy$

**47.** $\frac{y}{x^2 + y^2} = 3 + x^2$

**48–55** Find $z_x$ and $z_y$, where $z$ is defined implicitly by the given equation.

**48.** $x^2 + y^2 + z^2 = 1$

**49.** $xyz = e^{x+y+z}$

**50.** $xy - 3y^3 - 4xz^2 = 1$

**51.** $x^2 z^3 + xy = \sin(yz)$

**52.** $x\sin y + y^2 z - e^{xyz} = 1$

**53.** $e^x \sin y - 2z^2 + \frac{yz^2}{2} = 2$

**54.** $x^2 y + z^2 + y\ln z = 4$

**55.** $\ln(x^2 + y^2 + z^2) = 5 - xyz$

**56.** If $F(x,y) = 0$ implicitly defines $y$ as a function of $x$ and both $F$ and $y$ are twice-differentiable, show that

$$y''(x) = -\frac{F_{xx}(F_y)^2 - 2F_x F_y F_{xy} + F_{yy}(F_x)^2}{(F_y)^3}.$$

**57.** Use a tree diagram to write out the Chain Rule for the first partial derivatives $f_x$ and $f_y$ of $f(t,u,v,w)$, where $t = t(x,y)$, $u = u(x,y)$, $v = v(x,y)$, and $w = w(x,y)$.

**58.** If $f(x,y)$ is differentiable, where $x(u,v) = u + v$ and $y(u,v) = u - v$, prove the following.

$$\left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$$

**59.** Suppose $F(x,y,z)$ is differentiable, has nonzero first partial derivatives, and that $F(x,y,z) = 0$ defines each variable as a function of the other two variables (i.e., $x = x(y,z)$, $y = y(x,z)$, and $z = z(x,y)$). Prove the following equation.

$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x} = -1$$

**60–63** A function $f$ is said to be *homogeneous of degree n* if

$$f(tx,ty) = t^n f(x,y)$$

for all $n$, $t \in \mathbb{R}$. In these exercises, you will work with homogeneous functions. To begin, show that the given function in Exercises 60 and 61 is homogeneous and state the degree of homogeneity.

**60.** $f(x,y) = 5x^2 y^2 - 3x^3 y$

**61.** $f(x,y) = \frac{xy^2 + 2x^2 y}{\sqrt{x^4 + y^4}}$

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**62.** Let $f(x,y)$ be homogeneous of degree $n$. Prove the following formula.

$$x \cdot \frac{\partial f(x,y)}{\partial x} + y \cdot \frac{\partial f(x,y)}{\partial y} = nf(x,y)$$

(**Hint:** Consider the equation of homogeneity; differentiate both sides with respect to $t$, and let $t = 1$.)

**63.** Let $f(x,y)$ be homogeneous, as in Exercise 62. Prove the following formulas.

$$\frac{\partial f(tx,ty)}{\partial x} = t^{n-1} \cdot \frac{\partial f(x,y)}{\partial x} \quad \text{and}$$
$$\frac{\partial f(tx,ty)}{\partial y} = t^{n-1} \cdot \frac{\partial f(x,y)}{\partial y}$$

**64.** An ice "cube" in the form of a rectangular prism with a square base is melting so that the edge of the base is shrinking at $0.5 \text{ mm/min}$ while the height is decreasing at $0.75 \text{ mm/min}$. Determine the rate of change of its volume and surface area when the edge of the base is 20 mm and the height is 30 mm.

**65.** Consider a circular sector of radius $r$ and central angle $\theta$. Suppose that $\theta$ is increasing at a rate of 0.1 radians per minute, while $r$ is decreasing at a rate of 0.2 inches per minute. Find the rate of change of the area at the instant when $\theta = 1$ radian and $r = 15$ inches.

**66.** Suppose the height of a right circular cylinder is increasing at 1 millimeter per second. Determine the rate of change of the radius of the cylinder if the instantaneous rate of change of its volume is 0 when the radius is 50 millimeters and the height is 100 millimeters.

**67.** Consider a sand cone such as one formed by a child pouring sand out of a bucket. Assume that its height is growing at a rate of 0.1 inches per second, while its radius at 0.05 inches per second, at the instant when its height is 4 inches and its radius is 6 inches. Find the rate of change of the volume of the sand cone at this instant.

**68.** Find the rate of change of the lateral surface area of the sand cone at the instant described in Exercise 67.

**69.** Suppose that at a certain moment during takeoff, a plane's speed is $100 \text{ m/s}$, its acceleration $3 \text{ m/s}^2$, while its mass of 63,350 kg is decreasing at a rate of $1.15 \text{ kg/s}$ due to fuel consumption. Find the rate of change of the plane's kinetic energy at this instant.

**70.** Suppose the temperature of two moles of an ideal gas in a 50-liter (L) container is 323 kelvins (K) and increasing at a rate of $0.2 \text{ K/s}$, when at the same time, the volume of the container is increasing at a rate of $0.05 \text{ L/s}$. Find the rate of change of pressure at this instant. (For a refresher on the Ideal Gas Law, see Exercise 97 in Section 3.4.)

**71.*** Consider the insulated plane of Exercise 93 of Section 13.3, with temperature measured in degrees Celsius and time in minutes. Suppose a point is moving along the line $y = -x + \pi/2$, in the southeastern direction, at a speed of $\sqrt{2}/2$ unit lengths per minute. Supposing that it is at the point $(\pi/4, \pi/4)$ at $t = 5$ seconds, find the rate of temperature change from the moving point's perspective at that instant. (**Hint:** Determine the rates $dx/dt$, $dy/dt$, and use the Chain Rule.)

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