Proof

For convenience, we will prove the theorem in the two-variable case.

If f is differentiable at (a,b), then by definition the increment Δf can be written in such a way that

$$\begin{aligned}\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[f(a+\Delta x,b+\Delta y)-f(a,b)\right] &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y\right]\\ &= f_x(a,b)\lim_{\Delta x\to 0}\Delta x+f_y(a,b)\lim_{\Delta y\to 0}\Delta y\\ &\quad+\lim_{\Delta x\to 0}\varepsilon_1\lim_{\Delta x\to 0}\Delta x+\lim_{\Delta y\to 0}\varepsilon_2\lim_{\Delta y\to 0}\Delta y\\ &= 0.\end{aligned}$$

That is, $f(a+\Delta x,b+\Delta y)\to f(a,b)$ as $\Delta x\to 0$ and $\Delta y\to 0$.

By implication, then, the function in Example 9 does not meet the conditions of the Increment Theorem of Differentiability at $(0,0)$. In Exercise 103, you will show that this is indeed the case.

13.3 Exercises

1–4 Determine z_x and z_y at the indicated point; then find equations for the corresponding tangent lines that are parallel to, respectively, the *xz*-plane and the *yz*-plane.

1. $z=x+3xy^3;\quad (2,1)$
2. $z=x^4y-4xy^3;\quad (2,-1)$
3. $z=x(2-xy)^2;\quad (-1,0)$
4. $z=(xy^2+3)(x-3y);\quad (3,1)$

5–28 Find all first-order partial derivatives of the given function.

5. $f(x,y)=x^3+y^3$
6. $g(x,y)=xy^2-2xy$
7. $h(x,y)=5x^3y+y^4$
8. $r(x,y)=x^4-2x^2y^2+3y^6$
9. $V(r,h)=\dfrac{\pi r^2h}{3}$
10. $s(x,y)=(x^2+y^4)^5$
11. $k(x,y)=xy^2+\sqrt{2xy}$
12. $l(x,y)=(4xy-3)(x^2+1)$
13. $m(x,y)=(xy-3x^2)^4$
14. $n(x,y)=x(1-\sqrt{x}y)(y^2+2)$
15. $p(x,y)=\dfrac{x^2y}{x+y}$
16. $F(x,y)=\sqrt{1-x^2-y^2}$
17. $G(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$
18. $H(x,y)=e^{x^2y}$
19. $I(x,y)=\sin\dfrac{x}{y}$
20. $J(x,y)=y\ln\sqrt{x^2+y^2}$
21. $K(x,y)=\dfrac{2y^2}{x}-\dfrac{x^2}{2y}$
22. $L(x,y)=e^x\cos(xy)$
23. $M(x,y)=\arctan\sqrt{xy}$
24. $f(x,y,z)=e^xy^2\sin z$
25. $g(x,y,z)=x^{y/z}$
26. $A(a,b,c)=2(ab+ac+bc)$
27. $h(x,y,z)=x\cos(y+z^2)$
28. $w(r,s,t)=(r^2+2s^2+3t^2)^{3/2}$

29–34 Find a vector equation for *L*, the line tangent to the surface $z^2-4x^2-5y^2=0$ of Example 3 at the given point and parallel to the indicated coordinate plane.

29. $(1,1,3)$; the *xz*-plane
30. $(1,1,-3)$; the *yz*-plane
31. $\left(2,\dfrac{3}{\sqrt{5}},5\right)$; the *xz*-plane
32. $\left(2,\dfrac{3}{\sqrt{5}},-5\right)$; the *yz*-plane

33. $(-1,1,3)$; the xz-plane

34. $(-1,1,-3)$; the yz-plane

35–37 Use implicit differentiation to determine z_x and z_y.

35. $z^2 + 2xy - yz = 0$

36. $zx - (z+x)^2 = 3y$

37. $xz + \ln z - x^2 y = 0$

38. Treating y and z as independent variables, determine x_y and x_z from Exercise 36.

39. Find y_x and y_z by implicitly differentiating $y^2 - xy - z\ln y = 5$.

40. Recall the Ideal Gas Law, the equation relating the pressure P, volume V, and temperature T of an ideal gas: $PV = nRT$ (Section 3.4, Exercise 97). Assuming that n (i.e., the amount of gas in moles) is constant, differentiate implicitly to find the following partial derivatives and explain their physical meaning.

a. $\dfrac{\partial P}{\partial T}$ **b.** $\dfrac{\partial P}{\partial V}$ **c.** $\dfrac{\partial V}{\partial T}$

41–48 Verify the equality of the mixed partials f_{xy} and f_{yx}.

41. $f(x,y) = x^3 y - 2y^2 + 5xy^4$

42. $f(x,y) = (2y-x)^4$

43. $f(x,y) = (x^4 + y^4)^8$

44. $f(x,y) = xy^4 + 2y^{2/3}$

45. $f(x,y) = x^2 y^2 e^{xy}$

46. $f(x,y) = \ln(x^2 + y^2)$

47. $f(x,y) = e^{\sqrt{x^2+y^2}}$

48. $f(x,y) = 2y\cos(3x+4y)$

49–54 Verify that the third-order mixed partials g_{xyz}, g_{yzx}, and g_{zxy} are equal.

49. $g(x,y,z) = x^3 + 3yz^2 - xy^2 + 3z^3$

50. $g(x,y,z) = 2x(y-3z)^3$

51. $g(x,y,z) = x^3 y^3 z^3$

52. $g(x,y,z) = \sin(xyz)$

53. $g(x,y,z) = e^{xy}\cos z$

54. $g(x,y,z) = \dfrac{y}{x^2 + z^2}$

55. Find the partial derivative f_{xyy} of the function

$$f(x,y) = x\left(\frac{y^3}{3} + \ln y\right) + y(\cos x + x\sin x) + x\sec^2 x$$

by judiciously choosing the order of differentiation. (**Hint:** See Example 7.)

56–63 Use the most convenient order of differentiation to find the indicated partial derivative, as in Exercise 55.

56. g_{xzyz}; $g(x,y,z) = \ln(x^2 y^2) + z\cos(y^2) + z^2 x^3 y^4$

57. h_{rstu}; $h(r,s,t,u) = s\left(u^2 e^r + \dfrac{tr^2}{u}\right) + re^{rs}\sin t + t(s^2 + t\ln u)$

58. f_{xy}; $f(x,y) = xe^{y^2} + \dfrac{\ln x}{\sqrt{x}}$

59. f_{yxyx}; $f(x,y) = \sin(x^2 - y)$

60. f_{xyz}; $f(x,y,z) = z(xy^2 + \sqrt{x}\ln x) + x^2\sin y$

61. f_{xyz}; $f(x,y,z) = \dfrac{x^2 y}{2}(1+3z^2) + \cos y - \cos x$

62. f_{utst}; $f(r,s,t,u) = ru(t^2\sin s + 1) + \dfrac{e^{(t^2+1)}}{r}$

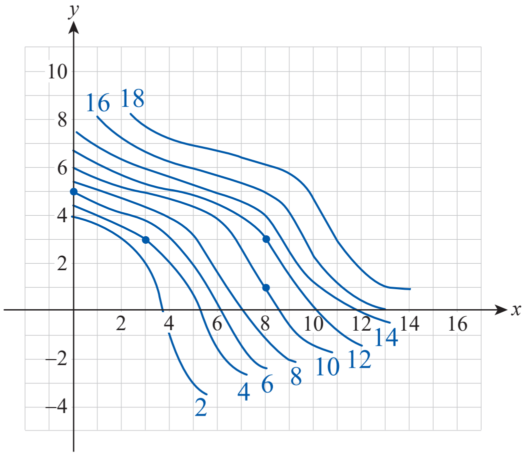
63. f_{xyzw}; $f(w,x,y,z) = \sqrt{w^2 + x^2 + y^2 + z^2}$

64–65 Use the definition of the partial derivative to find $\partial f/\partial x$ and $\partial f/\partial y$ for the given function.

64. $f(x,y) = x^2 y$ **65.** $f(x,y) = \dfrac{\sqrt{y}}{x}$

66. Use the contour map below to estimate the values of f_x and f_y at the indicated points. (Answers will vary.)

a. $(0,5)$ **b.** $(3,3)$ **c.** $(8,1)$ **d.** $(8,3)$



67–70 If possible, find a function that has the indicated partial derivatives. If such a function doesn't exist, explain why.

67. $f_x(x,y) = 2xy, \quad f_y(x,y) = x^2 - \cos y$

68. $f_x(x,y) = \frac{1}{\sqrt{x}} + \ln(y^2+1), \quad f_y(x,y) = \frac{2xy}{y^2+1}$

69. $f_x(x,y) = x - y^2, \quad f_y(x,y) = \sqrt{x} - 2y$

70. $f_x(x,y) = y\sin x + \frac{1}{2\sqrt{x}}, \quad f_y(x,y) = \cos x + 3y^2$

71. Recall the lens equation from Exercise 44 of Section 3.8:

$$\frac{1}{o} + \frac{1}{i} = \frac{1}{f},$$

where o is the object distance, i is the image distance, and f is the focal length of the lens. Find the partial derivative $\partial i/\partial o$ and explain its physical meaning.

72. Explain why there is a difference in sign between the first two formulas for $\Delta\rho$ in Example 4.

73. Recall that the kinetic energy E of an object of mass m moving at speed v is found from the formula $E = \frac{1}{2}mv^2$. Suppose that a 1 kg mass is moving at a speed of 5 m/s, but due to inaccuracies of the measuring devices and human error, its mass was recorded as 1002 grams and its speed was clocked at 495 cm/s. Estimate the relative error this causes in the value of E. (**Hint:** See Example 4.)

74. Consider the lens equation of Exercise 71, and suppose that a lens that is thought to be a 50 mm lens has an actual focal length of 52.56 mm. If, in addition, the actual object distance of 2.03 m is erroneously measured to be exactly 2 m, estimate the relative error all of this causes in the value of i.

75–80 Show that the given function satisfies the wave equation of Example 8.

75. $u(x,t) = \sin(x+ct) + e^{x-ct}$

76. $u(x,t) = \cosh(x+ct)$

77. $u(x,t) = \sin(\omega x)\cos(\omega ct)$

78. $u(x,t) = (x+ct)^4 + (x-ct)^4$

79. $u(x,t) = \cos(x+ct) + \frac{1}{x-ct}$

80. $u(x,t) = \ln(x+ct) + \sqrt{x-ct}$

81–86 Verify that f satisfies the two-dimensional form of Laplace's equation: $f_{xx} + f_{yy} = 0$.

81. $f(x,y) = e^x \cos y$

82. $f(x,y) = \sinh y \cos x$

83. $f(x,y) = x^2 - y^2$

84. $f(x,y) = e^x \sin y + e^y \cos x$

85. $f(x,y) = \arctan\frac{x}{y}$

86. $f(x,y) = \ln(x^2+y^2)^2$

87–89 Verify that the three-variable function satisfies Laplace's equation: $f_{xx} + f_{yy} + f_{zz} = 0$.

87. $f(x,y,z) = kxyz$

88. $f(x,y,z) = \sin(3x)e^{2y-\sqrt{5}z}$

89. $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

90–92 The temperature $u(x,t)$ of an insulated rod that is aligned with the x-axis satisfies the heat equation $u_t = \kappa u_{xx}$, where κ is a constant, called *thermal diffusivity*. Show that the functions in Exercises 90–92 satisfy the heat equation.

90. $u(x,t) = e^{-\kappa t}\sin x$ **91.** $u(x,t) = e^{-4\kappa t}\sin 2x$

92. $u(x,t) = 5e^{-\kappa n^2 t}\cos(nx)$

93. Suppose that the temperature of an insulated planar surface is $T(x,y,t) = e^{-5t}\sin x\cos 2y$. Find the rates of change of temperature in the x- and y-directions, respectively, at the point $(\pi/4, \pi/4, 0)$.

94. Show that $T(x,y,t)$ of Exercise 93 satisfies the two-dimensional heat equation $u_t = \kappa(u_{xx} + u_{yy})$ with $\kappa = 1$.

95. Generalizing Exercises 92 and 94, show that all functions of the form

$$u(x,y,t) = e^{-\kappa(m^2+n^2)t}\sin(mx)\cos(ny)$$

satisfy the two-dimensional heat equation ($m, n \in \mathbb{R}$).

96. Suppose a guitar string, originally aligned with the positive x-axis, is plucked and the function $g(x,t)$ describes the displacement of the point $(x,0)$ as a function of t. What can you say about a point $(x_0,0)$ at time $t = t_0$ if both partial derivatives $g_x(x_0,t_0)$ and $g_t(x_0,t_0)$ are positive? What if, in addition, $g_{xx}(x_0,t_0)$ and $g_{tt}(x_0,t_0)$ are both negative?

97.* Recall from trigonometry that the area of a circular sector of radius r and central angle α is $A = \frac{1}{2}r^2\alpha$. Denoting the circumference of the sector by C, prove that the area can also be expressed as $A = \frac{1}{2}Cr - r^2$; then determine $\partial A/\partial r$ from both of the given area formulas and explain why your answers are not equivalent.

98. Consider the general cubic polynomial in the variables x and y:

$$P(x,y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J$$

Find conditions on the coefficients to ensure that $P(x,y)$ satisfies Laplace's equation (such functions are called *harmonic*).

99.* If we invest P dollars and take inflation and taxes into consideration, the future value of our investment in n years is

$$A = P\left[\frac{1+r(1-T)}{1+I}\right]^n \text{ dollars,}$$

where I and T are the inflation and tax rates, respectively, and r is the annual interest rate. Suppose we invest \$15,000 for 10 years at a rate of 12%. Use the partial derivatives $\partial A/\partial I$ and $\partial A/\partial T$ to decide whether it is inflation or the tax rate that affects the investment more drastically.

100.* Prove that the first partial derivatives of a harmonic function are themselves harmonic, if they have continuous second partial derivatives (see Exercise 98 for the definition of harmonic functions).

101.* Let R denote the net resistance of two resistors R_1 and R_2 in a parallel circuit, which satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

a. Generalize the above equation to show that the net resistance of n resistors in a parallel circuit is as follows.

$$R = \frac{\prod_{i=1}^{n} R_i}{\sum_{i=1}^{n} \prod_{j \neq i} R_j}$$

b. For a fixed index k, find a formula for $\partial R/\partial R_k$. (**Hint:** It is helpful to look at the cases $n = 2$ and $n = 3$ before generalizing.)

102. Prove that if $u(x,t)$ can be written in the form $u(x,t) = u_1(x+ct) + u_2(x-ct)$ for some one-variable functions u_1 and u_2 that are at least twice differentiable, then $u(x,t)$ is a solution of the wave equation of Example 8.

103. Show that the function in Example 9 does not meet the conditions of the Increment Theorem of Differentiability at $(0,0)$. (**Hint:** Show, for example, that f_x is not continuous at the origin by looking at f_x at nearby points on the x-axis.)

104–107 Decide whether the given function is differentiable at the origin. Give a reason for your answer.

104. $f(x,y) = y^2e^x - x^2y$

105. $f(x,y) = \sqrt{x^2 + 2y^2}$

106. $f(x,y) = \dfrac{1}{x^2 + y^2 + 1}$

107. $f(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

108–113 Show that f is not differentiable at the origin, even though both $f_x(0,0)$ and $f_y(0,0)$ exist. (In Exercises 112 and 113, generalize to the three-variable case.)

108. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

109. $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

110. $f(x,y) = \begin{cases} \dfrac{-4x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

111. $f(x,y) = \begin{cases} 1 & \text{if } y^2 < x < 2y^2 \\ -1 & \text{otherwise} \end{cases}$

112. $f(x,y,z) = \begin{cases} 2 & \text{if } xyz = 0 \\ -2 & \text{if } xyz \neq 0 \end{cases}$

(**Hint:** See Example 9.)

113. $f(x,y,z) = \begin{cases} 1 & \text{if } x^2 + y^2 < z < 2x^2 + 2y^2 \\ -1 & \text{otherwise} \end{cases}$

(**Hint:** See Exercise 111.)

114.* Consider the given piecewise-defined function.

$$F(x,y)=\begin{cases}\dfrac{xy^2}{2(x^2+y^2)} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

a. In contrast to some of the previous exercises, show that F is continuous at $(0,0)$. (**Hint:** See, for example, Exercise 31 or 41 of Section 13.2.)

b. Prove that both partial derivatives of F are equal to 0 at $(0,0)$. (**Hint:** Use the definition of partial derivatives.)

c. Prove that F is not differentiable at $(0,0)$. (**Hint:** As noted after Example 9 in the text, the derivative of F at $(0,0)$ must take into account *all* the different limiting approaches to $(0,0)$; however, compare the common value of the two partial derivatives of F at $(0,0)$ with $\lim\limits_{h\to 0}\dfrac{F(h,h)-F(0,0)}{h}$.)

115–118 For certain functions, it is fairly straightforward to demonstrate differentiability by using the definition. For example, let $f(x,y)=x+y^2$ and note the following.

$$\begin{aligned}\Delta f(x,y)&=f(x+\Delta x,y+\Delta y)-f(x,y)\\&=x+\Delta x+(y+\Delta y)^2-(x+y^2)\\&=x+\Delta x+y^2+2y\Delta y+(\Delta y)^2-x-y^2\\&=\Delta x+2y\Delta y+(\Delta y)^2\\&=f_x(x,y)\Delta x+f_y(x,y)\Delta y+0\cdot\Delta x+\Delta y\cdot\Delta y\end{aligned}$$

Letting $\varepsilon_1=0$ and $\varepsilon_2=\Delta y$, we see that ε_1 and ε_2 approach 0 as both Δx and Δy approach 0, as needed. In Exercises 115–118, mimic this process of using the definition to prove that the function is differentiable.

115. $f(x,y)=x^2-2y$ **116.** $g(x,y)=xy^2$

117. $h(x,y)=2(x^2+y^2)$

118. $k(x,y)=x^3-4x+3y$

119. Find all points (x,y) where the function $f(x,y)=|x-y|$ is differentiable.

120. Repeat Exercise 119 for the function $g(x,y)=\sqrt{x^2+y^2}$.