13.2 Exercises

1–4 Use the limit laws to find the indicated limits, assuming that $\lim_{(x,y)\to(a,b)} f(x,y)=4$ and $\lim_{(x,y)\to(a,b)} g(x,y)=-1$.

1. $\lim_{(x,y)\to(a,b)}\left[f(x,y)-2g(x,y)\right]$

2. $\lim_{(x,y)\to(a,b)}\left[5f(x,y)g(x,y)\right]$

3. $\lim_{(x,y)\to(a,b)}\left[\frac{4g(x,y)}{f(x,y)}+f(x,y)\right]$

4. $\lim_{(x,y)\to(a,b)}\frac{3f(x,y)+g(x,y)}{g(x,y)}$

5–22 Determine whether the indicated limit exists. If so, find it.

5. $\lim_{(x,y)\to(0,0)}\frac{x\sqrt{y}-y^{3/2}}{\sqrt{x}+\sqrt{y}}$ **6.** $\lim_{(x,y)\to(1,2)}\frac{\sqrt{x}-\sqrt{y-1}}{x^2-xy+x}$

7. $\lim_{(x,y)\to(0,0)}\frac{(x+1)\left(\sqrt{y+5}-\sqrt{5}\right)}{xy+y}$

8. $\lim_{(x,y)\to(0,0)}\left(2x-y^2\right)$

9. $\lim_{(x,y)\to(2,-1)}\frac{x+2y}{y+3x^2}$ **10.** $\lim_{(x,y)\to(1,4)}\left(2xy^2+5x^4\sqrt{y}\right)$

11. $\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$ **12.** $\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^2}$

13. $\lim_{(x,y)\to(0,0)}\frac{(1-\cos x)y}{x^4+y^4}$ **14.** $\lim_{(x,y)\to(0,0)}\frac{x-y}{x^3-y^3}$

15. $\lim_{(x,y)\to(0,0)}\frac{x^3-y^3}{x-y}$ **16.** $\lim_{(x,y)\to(1/2,2)}\arctan\left(\frac{xy}{y-1}\right)$

17. $\lim_{(x,y)\to(0,0)}\frac{2\sqrt{x}y^3}{x^3+y^4}$ **18.** $\lim_{(x,y)\to(1,0)}\frac{e^{xy}+e^{-\sqrt{y}}}{2xy+1}$

19. $\lim_{(x,y)\to(0,0)}\sin\frac{2}{x^2+y^2}$ **20.** $\lim_{(x,y)\to(0,0)}\frac{2\sin\left(x^2+y^2\right)}{x^2+y^2}$

21. $\lim_{(x,y,z)\to(0,0,0)}\frac{x^2-3y^2+4z^2}{x^2+3y^2+4z^2}$

22. $\lim_{(x,y,z)\to(0,0,0)}\frac{5xy^4}{x^4+y^4+z^4}$

23. Find the domains of the functions in Exercises 5 and 6.

24. With reference to the function of Example 1,

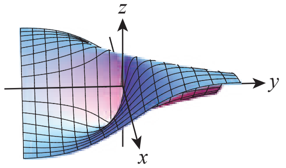
$$f(x,y)=\frac{x^2-y^2}{x^2+y^2},$$

show that if the origin is being approached in the xy-plane along the lines $y=mx$ for various values of m, all limiting values between -1 and 1 are possible.

25. Recall the following function introduced in the conclusion of Example 3:

$$g(x,y)=\frac{5xy}{x^2+y^2}.$$

Show that $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. (**Hint:** See Exercise 24.)



26. Show that the limit of the function

$$f(x,y)=\frac{x^4y^4}{\left(x^2+y^4\right)^3}$$

does not exist at $(0,0)$. (**Hint:** In addition to the line $y=x$, consider the limit along the curve $y^2=x$. Explain why it would not be helpful to restrict the paths of approach to straight lines through the origin.)

27. Show that the function

$$g(x,y)=\frac{2x^{3/2}y}{x^3+y^2}$$

has a limiting value of 0 when the origin is approached in the xy-plane along any parabola $y=cx^2$, but any limiting value between -1 and 1 can be achieved by considering the curves $y=cx^{3/2}$.

28. Prove directly that the limit in Example 3 is 0. (**Hint:** First argue that if $x=0$, the function is 0 on its domain, and in the case of $x\neq 0$, divide the numerator and denominator by x^2 to obtain an expression whose limit is 0 as $y\to 0$.)

29. Use the Squeeze Theorem to give a new proof for Exercise 28. (**Hint:** Notice that $0\le\left|\frac{x^2}{x^2+y^2}\right|\le 1$, and multiply this inequality by $|5y|$; then apply the Squeeze Theorem.)

30. Prove or disprove: $\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}=0.$

31. Prove that for $h(x,y)=5xy/\sqrt{x^2+y^2}$, we have $\lim_{(x,y)\to(0,0)} h(x,y)=0$, and contrast this with $g(x,y)$ of Exercise 25. (**Hint:** Rewrite $h(x,y)$ using polar coordinates, so that $x=r\cos\theta$ and $y=r\sin\theta$ and see what happens as $r\to 0$.)

32. The following function looks algebraically similar to $f(x,y)$ of Example 1.

$$k(x,y)=\frac{x^3-y^3}{x^2+y^2}$$

Show that $\lim_{(x,y)\to(0,0)} k(x,y)=0$. (See the hint given in Exercise 31.)

33. Consider the polar coordinate approach of Exercise 31 and use it to find a new proof of the fact that the limit in Example 3 is 0.

34.* Find three different proofs of the fact that

$$\lim_{(x,y)\to(0,0)} \frac{x^2\ln(x+1)}{x^2+y^2}=0.$$

(**Hint:** Use a direct proof, then the Squeeze Theorem, and finally a polar coordinate approach.)

35.* Use an ε-δ argument to show that

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{2(x^2+y^2)}=0.$$

36–39 Prove that the indicated limit exists, and find its value.

36. $\lim_{(x,y)\to(0,0)} \frac{4y^2\sin x}{x^2+4y^2}$

37. $\lim_{(x,y)\to(0,0)} \frac{x^4-3x^2-9y^2}{x^2+3y^2}$

38. $\lim_{(x,y)\to(0,0)} \frac{4xy-\sqrt{4x^2+4y^2}}{\sqrt{x^2+y^2}}$

39. $\lim_{(x,y,z)\to(0,0,0)} \frac{2y^5}{x^4+y^4+z^4}$

40. Is there a value of a so that $\lim_{(x,y)\to(a,a-1)} \frac{x}{y-x+1}$ exists? Explain.

41. For $\alpha,\beta\geq 0$, prove that $\lim_{(x,y)\to(0,0)} \frac{x^\alpha y^\beta}{x^2+y^2}=0$ if $\alpha+\beta>2$; otherwise the limit does not exist. Can you generalize this result? Use it to revisit Exercises 28, 30, and 35.

42. Use an ε-δ argument to show that $\lim_{(x,y)\to(a,b)} (x+y)=a+b$.

43. Suppose that $\lim_{(x,y)\to(a,b)} f(x,y)=L$ and $L>0$. Prove that there exists a δ-neighborhood of (a,b) such that $f(x,y)$ is positive for every point (x,y) in that neighborhood.

44–53 Determine where the given function is continuous.

44. $f(x,y)=\frac{2}{1+x^2+y^2}$

45. $g(x,y)=\frac{3y^2}{(x^2+3)(y^2+3)}$

46. $h(x,y)=\cot^{-1}\left(\frac{2y}{\sqrt{x+y}}\right)$

47. $r(x,y)=\sqrt{e^{xy}}$

48. $s(x,y)=5\sqrt{x}y^2+\frac{3}{\ln(x^2y^2)}$

49. $k(x,y)=e^{\arctan(2x^2y)}$

50. $q(x,y)=\frac{x^2}{\sqrt{4-x^2-y^2}}$

51. $m(x,y,z)=\ln(3z-x^2-2y^2)$

52. $n(x,y,z)=\sqrt{z^2-x^2-2y^2}$

53. $p(x,y,z)$

$$=\begin{cases}\frac{1-\cos\sqrt{9-x^2-y^2-z^2}}{\sqrt{9-x^2-y^2-z^2}} & \text{if } x^2+y^2+z^2<9\\ 0 & \text{if } x^2+y^2+z^2=9\end{cases}$$

54–59 Find any discontinuities of the given function and classify them as removable or nonremovable.

54. $f(x,y)=\frac{(x-1)y^2}{(x-1)^2+y^2}$

55. $g(x,y)=\frac{xy}{x^2+y^2}$

56. $h(x,y)=\frac{x}{x^2+y^2}$

57. $p(x,y)=\ln\left(\frac{1}{x^2+y^2}\right)$

58. $m(x,y)=\frac{x}{\sqrt{x^2+(y-2)^2}}$

59. $k(x,y,z)=\frac{x^2-2y^2+z^2}{x^2+y^2+z^2}$

Concept Check

60–65 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

60. If $\lim_{(0,y)\to(0,0)} f(x,y) = L$ and $\lim_{(x,0)\to(0,0)} f(x,y) = L$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

61. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and its value is L, then $f(a,b) = L$.

62. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and its value is L, then $\lim_{x\to a} f(x,b) = L$.

63. If $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and the limiting value of $f(x,y)$ along the line $y = x$ is $f(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.

64. If $f(x,y) < 0$ in some ε-neighborhood $B_\varepsilon(0,0)$ of the origin and $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, then $\lim_{(x,y)\to(0,0)} f(x,y) < 0$.

65. If $f(0,0) = L$ and $f(x,y)$ is continuous for all $(x,y) \neq (0,0)$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.