12.4 Exercises

1–6 Describe in terms of **u** and $\mathbf{u}_\perp$ the velocity and acceleration vectors of a particle with the given position function.

1. $\mathbf{r}(t) = \langle 2t\cos 2t, 2t\sin 2t\rangle$
2. $\mathbf{r}(t) = \langle (t^2+1)\cos 5t, (t^2+1)\sin 5t\rangle$
3. $\mathbf{r}(t) = \langle \sqrt{t}\cos(4t+1), \sqrt{t}\sin(4t+1)\rangle$
4. $\mathbf{r}(t) = \left\langle e^t\cos\frac{t}{2}, e^t\sin\frac{t}{2}\right\rangle$
5. $\mathbf{r}(t) = \langle 2\sin 2t\cos(t^3), 2\sin 2t\sin(t^3)\rangle$
6. $\mathbf{r}(t) = \langle a(\cos t-1)\cos bt, a(\cos t-1)\sin bt\rangle$

7. Prove that if a satellite or planet is moving in a circular orbit, then its speed is constant. (**Hint:** Use the fact that $\mathbf{C}_1$ is constant, an observation made in the proof of Kepler's First Law.)

8. Prove that in order for a moon or satellite to stay in a circular orbit of radius R around a planet of mass M, the required orbital speed is $v = \sqrt{GM/R}$. (**Hint:** Recall that v is constant by Exercise 7, thus the magnitude of acceleration is $a = v^2/R$. Use this and Newton's Second Law to finish the proof. Alternatively, use the last equation in our proof of Kepler's Third Law, noting that $a = R$.)

9. A satellite is in a circular orbit 219.2 kilometers above Earth's surface. Use Exercise 8 to find its orbital speed. Express your answer in kilometers per hour. (Approximate the radius of Earth by 6371 kilometers and its mass by 5.9736×10^{24} kilograms. Recall that $G \approx 6.6738\times10^{-11}\ \mathrm{Nm}^2/\mathrm{kg}^2$.)

10. Given that the perihelion of Earth's orbit is approximately 147,098,290 km, with its *aphelion* (the distance farthest from the sun) being 152,098,232 km, and using its period of T = 365.256 days, estimate the mass of the sun. (**Hint:** Find the length of the orbit's semimajor axis first, then use the last equation in our derivation of Kepler's Third Law. For the value of G, see Exercise 9.)

11. The period of one revolution of the moon around Earth is approximately 27.3217 days, its *perigee* (distance from Earth upon closest approach) is approximately 361,400 km, while its *apogee* (its greatest distance from Earth) is about 405,000 km. Use these data to estimate the mass of Earth.

12. Use the data given in Exercise 11, along with Kepler's Third Law, to estimate the necessary height above Earth's surface for a *geostationary satellite*. (A geostationary satellite is one that is in a near-circular orbit over the equator, orbiting in the direction of Earth's rotation with a period of 24 hours, thus appearing stationary for an observer on the ground. **Hint:** While you may compute the height directly, an easier approach suggested by the problem is to compare the satellite's orbit with that of the moon and use Kepler's Third Law.)

13. Use Kepler's Third Law, along with Earth's orbital data given in Exercise 10, and Mars' period of 686.971 (Earth) days to estimate the semimajor axis of Mars' orbit.

14. The length of the semimajor axis of Neptune's orbit is 30.0476 *astronomical units* (1 astronomical unit, abbreviated AU, is equal to 149,597,870.700 km, which is approximately the mean distance between Earth and the sun). Estimate the period of Neptune in Earth years. (For orbital data on Earth, see Exercise 10.)

15. Prove that the ratio of the perihelion and aphelion of a planet is equal to the inverse ratio of its speeds at the perihelion and aphelion positions. (**Hint:** Use Kepler's Second Law, namely, the equation that says $dA/dt = \frac{1}{2}r^2\theta' = C_1/2$ is constant.)

16. Show that the minimum distance of a moon or satellite from the planet it is orbiting (the perigee) is $r_p = a(1-e)$, while the maximum distance (the apogee) is $r_a = a(1+e)$, where e is the eccentricity of the orbit. (**Hint:** In order to express ed in terms of a and b in the polar equation of the planet, see the proof of Kepler's Third Law in the text.)

17. Use Exercise 16 along with the orbital data of Earth given in Exercise 10 to find the eccentricity of Earth's orbit, then write a polar equation of the orbit (with the sun at the origin).

18. Find the aphelion of the orbit of Halley's comet and write a polar equation for the orbit given that $e \approx 0.967$ and $a \approx 17.94$ astronomical units (AU). (See Exercise 16. Use the unit AU in your answer.)

19. Find how much time passes between two consecutive visits of Halley's comet to the solar system. (See Exercises 14 and 18.)

20. Define $r_0 = |\mathbf{r}(0)|$ and $v_0 = |\mathbf{v}(0)|$, and show that with this notation,

$$A(t) = \frac{r_0 v_0}{2} t.$$

(**Hint:** Note that $v_0 = r_0 \theta'(0)$ since $r'(0) = 0$.)

21. Modify your proof of Exercise 20 to show that the equality $v = \dfrac{2\pi ab}{rT}$ holds at perigee or apogee for an orbiting planet or satellite.

22. If v_p and v_a denote a planet's speeds at perigee and apogee, respectively, prove that

$$v_p(1-e) = v_a(1+e).$$

(**Hint:** As in Exercise 20, note that $v = r\theta'$ at perigee and apogee, and that $r^2\theta' = C_1$ is a constant. Use this latter equation for both perigee and apogee positions, along with Exercise 16.)

23. Use Exercise 21 and Earth's orbital data given in Exercise 10 to find the speeds of Earth at perihelion and aphelion, respectively. Express your answer in kilometers per second, and then convert it to miles per hour.

24. Repeat Exercise 23 for Mars, if its perihelion and aphelion are 2.0662×10^8 kilometers and 2.4923×10^8 kilometers, respectively, with its period being 1.88079 years.

25. Use Exercises 11 and 21 to find the moon's speed at perigee and apogee, respectively. Express your answer in kilometers per second, then convert it to miles per hour.

26. Repeat Exercise 25 for Jupiter's moon Europa, if given that the eccentricity of its orbit is 0.0101, the length of its semimajor axis is 671,100 kilometers, and its period is 3.5512 days.

27. Repeat Exercise 26 for Halley's comet. Express your answer in the following units: astronomical units per day, kilometers per second, and miles per hour. (For orbital data, see Exercises 18 and 19.)

28.* Suppose a moon or planet is orbiting another planet (or star) of mass M. As before, let r_p denote the perigee or perihelion, while v_p is the speed at perigee or perihelion, as applicable. Prove that the orbit can be classified as an ellipse, a parabola, or a hyperbola according to the values of r_p and v_p as follows:

if $r_p v_p^2 = GM$, the orbit is a circle;

if $GM < r_p v_p^2 < 2GM$, the orbit is an ellipse;

if $r_p v_p^2 = 2GM$, the orbit is a parabola;

if $r_p v_p^2 > 2GM$, the orbit is a hyperbola.

(The last two orbit types are called *open*; such orbits are exhibited by comets entering the solar system once and then leaving it forever. **Hint:** Referring to the proof of Kepler's First Law in the text, since $e = \dfrac{C_2}{GM}$, if you show that $\dfrac{C_2}{GM} = \dfrac{r_p v_p^2}{GM} - 1$, the conclusion will follow. To that end, it will suffice to show that $C_2 = r_p v_p^2 - GM$. From the proof of Kepler's First Law, convince yourself that $r_p(GM + C_2) = C_1^2$ and note from Exercise 22 that $C_1 = r_p^2\theta'$, which will finish the proof. Finally, we note that in the case of $C_1 = 0$, the moon or planet falls along a straight line into the star or planet it is orbiting.)

29. A novice astronaut is in an elliptical orbit around Earth. In an attempt to slow down to better behold the beautiful view, she plans to apply reverse thrust to decrease (tangential) speed. However, Mission Control advises her to check her calculations, for this may actually cause the craft to go around Earth faster. Who is correct and why? Use Kepler's Third Law to provide an explanation.

30.* Suppose the astronaut of Exercise 29 erroneously applied reverse thrust and ended up in a circular orbit of radius $R = 6600$ km, instead of a planned new orbit with perigee of $r_p = 6600$ km and apogee $r_a = 10{,}000$ km. At a time determined by Mission Control, she is instructed to apply forward thrust to enter into the desired elliptical orbit. If her thrusters lend the spacecraft an acceleration of 0.0205 km/s^2, how long does she have to burn them in order to accomplish this task? (**Hint:** By Exercise 8, the speed in the circular orbit is $v = \sqrt{GM/R}$. Next, note that the speed at perigee in the new orbit will be $v_p = C_1/r_p$. By determining the eccentricity of the planned orbit and recalling from the proof of Kepler's Third Law that $ed = C_1^2/GM = a(1-e^2)$, you can determine C_1, and then v_p. For data on Earth, see Exercises 9 and 10.)