Normal Component	$a_N = \dfrac{\lvert \mathbf{r}' \times \mathbf{r}'' \rvert}{\lvert \mathbf{r}' \rvert}$
Third Derivative	$\mathbf{r}''' = \left[s''' - \kappa^2 (s')^3 \right] \mathbf{T} + \left[3\kappa s' s'' + \kappa' (s')^2 \right] \mathbf{N} + \kappa\tau (s')^3 \mathbf{B}$
Curvature	$\kappa = \left\lvert \dfrac{d\mathbf{T}}{ds} \right\rvert = \dfrac{\lvert \mathbf{T}' \rvert}{\lvert \mathbf{r}' \rvert} = \dfrac{\lvert \mathbf{r}' \times \mathbf{r}'' \rvert}{\lvert \mathbf{r}' \rvert^3}$
Torsion	$\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\lvert \mathbf{r}' \times \mathbf{r}'' \rvert^2}$
Frenet-Serret Formulas	$\dfrac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$, $\dfrac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}$, and $\dfrac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$

Table 1

12.3 Exercises

1–10 Find the unit tangent, normal, and binormal vectors for the given curve.

1. $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$
2. $\mathbf{r}(t) = \langle \sin t, \cos t, \sqrt{3}t + 1 \rangle$
3. $\mathbf{r}(t) = \langle 3\sin 2t, 3\cos 2t, 3 \rangle$
4. $\mathbf{r}(t) = \langle t, t, 3t^2 \rangle$
5. $\mathbf{r}(t) = \langle \sin \pi t, \cos \pi t, \pi t \rangle$
6. $\mathbf{r}(t) = \langle 2t, t^3, t \rangle$
7. $\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle$
8. $\mathbf{r}(t) = \langle 4e^t \cos t, 4e^t \sin t, 1 \rangle$
9. $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^{t+1} \rangle$
10. $\mathbf{r}(t) = \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$

11–20 Use the results of Exercises 1–10 to determine equations for the osculating, normal, and rectifying planes associated with the curve at the indicated point.

11. $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$; $t = 0$
12. $\mathbf{r}(t) = \langle \sin t, \cos t, \sqrt{3}t + 1 \rangle$; $t = \dfrac{\pi}{2}$
13. $\mathbf{r}(t) = \langle 3\sin 2t, 3\cos 2t, 3 \rangle$; $t = 0$
14. $\mathbf{r}(t) = \langle t, t, 3t^2 \rangle$; $t = 1$
15. $\mathbf{r}(t) = \langle \sin \pi t, \cos \pi t, \pi t \rangle$; $t = \dfrac{1}{2}$
16. $\mathbf{r}(t) = \langle 2t, t^3, t \rangle$; $t = 2$
17. $\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle$; $t = \pi$
18. $\mathbf{r}(t) = \langle 4e^t \cos t, 4e^t \sin t, 1 \rangle$; $t = 0$
19. $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^{t+1} \rangle$; $t = \dfrac{\pi}{2}$
20. $\mathbf{r}(t) = \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$; $t = \dfrac{\pi}{4}$

21–28 Use Exercises 3–10 to find a parametric description of the surface.

21. The ribbon of width $\frac{1}{2}$ centered on $\mathbf{r}(t) = \langle 3\sin 2t, 3\cos 2t, 3 \rangle$
22. The ribbon of width 2 centered on $\mathbf{r}(t) = \langle t, t, 3t^2 \rangle$
23. The circular tube of radius 1 centered on $\mathbf{r}(t) = \langle \sin \pi t, \cos \pi t, \pi t \rangle$
24. The circular tube of radius $\frac{1}{3}$ centered on $\mathbf{r}(t) = \langle 2t, t^3, t \rangle$
25. The elliptical tube of major axis 0.6 (in the normal direction) and minor axis of 0.4 centered on $\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle$
26. The ribbon of width $\frac{1}{4}$ centered on $\mathbf{r}(t) = \langle 4e^t \cos t, 4e^t \sin t, 1 \rangle$
27. The hypocycloid $x = \cos t + 2\cos(t/2)$ and $y = \sin t - 2\sin(t/2)$, $-2\pi < t \le 2\pi$ wrapped around $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^{t+1} \rangle$
28. The circular tube of radius $\frac{1}{8}$ centered on $\mathbf{r}(t) = \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle$

29. Show how the fact that the unit tangent vector function $\mathbf{T}(t)$ corresponding to a smooth curve $\mathbf{r}(t)$ has constant length implies that $\mathbf{T}\cdot\mathbf{T}'=0$ for all t.

30. Show that if $\mathbf{r}(t)$ is a plane curve, then $\mathbf{N}(t)$ always points toward its "concave side," that is, "in the direction the curve bends."

31–33 Use your reparametrization results from Exercises 11, 13, and 14 of Section 12.2 to calculate the curvature and torsion functions for these curves.

31. $\mathbf{r}(t)=\langle 1+2t,3-5t,4+4t\rangle$

32. $\mathbf{r}(t)=\langle 2\cos t,2\sin t,3t\rangle$

33. $\mathbf{r}(t)=\langle t,\cosh t,\sinh t\rangle$

34. Find the curvature of the ellipse

$$\mathbf{r}(t)=\langle 4\cos t,3\sin t,0\rangle$$

at $t=\pi/2$. What about $t=0$?

35. Prove that the curvature of the helix $\mathbf{r}(t)=\langle a\cos t,a\sin t,bt\rangle$ is obtained by the following formula.

$$\kappa=\frac{a}{a^2+b^2}$$

What can you say if $b=0$?

36. Show that the torsion for the helix of Exercise 35 is $\tau=b/(a^2+b^2)$.

37. Generalize Exercises 35 and 36 to obtain formulas for the curvature and torsion functions of the general helix $\mathbf{r}(t)=\langle a\cos\omega t,a\sin\omega t,bt\rangle$.

38. Prove that if a curve $\mathbf{r}(t)$ lies in a plane, its torsion function is identically zero.

39–48 Calculate the curvature and torsion functions for the given curve. (**Hint:** Use the relevant formulas from the summary table.)

39. $\mathbf{r}(t)=\langle t,2\sin t,2\cos t\rangle$

40. $\mathbf{r}(t)=\langle t,3t+2,3t-1\rangle$

41. $\mathbf{r}(t)=\left\langle 1,t,\frac{1}{t}\right\rangle$

42. $\mathbf{r}(t)=\langle 1,t,e^t\rangle$

43. $\mathbf{r}(t)=\langle 2\cos 3t,2\sin 3t,1\rangle$

44. $\mathbf{r}(t)=\langle 1,t,t^2\rangle$

45. $\mathbf{r}(t)=\langle \sin 3t,\cos 3t,4t\rangle$

46. $\mathbf{r}(t)=\langle 2e^t\sin t,2e^t\cos t,1\rangle$

47. $\mathbf{r}(t)=\langle e^t,e^t\cos t,e^t\sin t\rangle$

48. $\mathbf{r}(t)=\langle t\cos t+\sin t,0,\cos t-t\sin t\rangle$

49. Prove that for the Bernoulli spiral of Exercise 85 in Section 12.1, the arc length $s(t)$ and curvature $\kappa(t)$ are inversely proportional. (**Note:** This is commonly interpreted as the arc length $s(t)$ and the radius of curvature being directly proportional; for the radius of curvature see the discussions preceding Exercises 60 and 74.)

50. If $f(x)$ is at least twice differentiable, prove that the curvature function of the plane curve $y=f(x)$ can be determined as

$$\kappa(x)=\frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}.$$

What can you conclude about the curvature of a plane curve at any of its inflection points? (**Hint:** Use $t=x$ as a parameter.)

51–53 Use Exercise 50 to evaluate the curvature of the plane curve at the indicated point.

51. $y=x^3;\quad x=1$

52. $y=\sin x;\quad x=\frac{\pi}{4}$

53. $y=\ln|\cos x|;\quad x=\frac{\pi}{6}$

54–56 Use Exercise 50 to find the point(s) of maximum curvature for the curve.

54. $f(x)=\frac{x^2}{2}$

55. $f(x)=\cos x$

56. $f(x)=e^x$

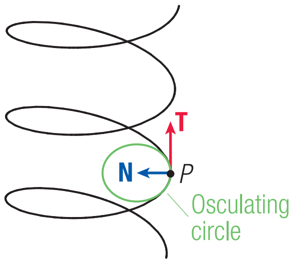
57. Generalize Exercise 50 for a plane curve $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ to obtain the formula

$$\kappa(t)=\frac{|x'(t)y''(t)-x''(t)y'(t)|}{\left([x'(t)]^2+[y'(t)]^2\right)^{3/2}}.$$

58. Use implicit differentiation along with Exercise 50 to find the curvature of $(6-x)y^2=2x^3$ at the point $(2,2)$. (Recall from Exercise 27 of Section 3.5 that this curve is a *cissoid*.)

59. Find the curvature of the cycloid $x=t-\sin t$, $y=1-\cos t$ at $t=\pi$.

60–68 Suppose the curvature of a curve $\mathbf{r}(t)$ at the point $P=\mathbf{r}(t_0)$ is nonzero. The **osculating circle** of the curve at P is the circle of radius $1/\kappa(t_0)$ that has the same tangent at P as $\mathbf{r}(t)$ does, and whose center lies in the direction of $\mathbf{N}(t_0)$ from P. (You can think of this as the "tangent circle." Note that it lies in the osculating plane, being tangent to $\mathbf{r}(t)$ so that both the circle's tangent line and curvature at P are the same as those of $\mathbf{r}(t)$. The osculating circle is rightfully called the "best-fitting circle" at P.)



Find the osculating circle of the graph of the equation at the indicated point.

60. $y=x^2;\quad x=0$

61. $y=x^2-1;\quad x=-\dfrac{1}{2}$

62. $y=\sqrt{x};\quad x=1$

63. $y=\cos x;\quad x=0$

64. $y=\cos x;\quad x=\dfrac{\pi}{4}$

65. $y=\dfrac{1}{x};\quad x=1$

66. $y=e^x;\quad x=0$

67. $y=x^3-x;\quad x=1$

68. $xy+2x+y=2;\quad x=1$

69–72 Parametrize the osculating circle of the curve at the indicated point. (**Hint:** For space curves, remember that the osculating circle lies in the plane spanned by **T** and **N**.)

69. $\mathbf{r}(t)=\langle t-\sin t, 1-\cos t\rangle;\quad t=\pi$
(See Exercise 59.)

70. $\mathbf{r}(t)=\langle \sin t, \cos t, 2t\rangle;\quad t=\pi$

71. $\mathbf{r}(t)=\left\langle t^2, t, \dfrac{t^3}{3}\right\rangle;\quad t=0$

72. $\mathbf{r}(t)=\left\langle 2\cos t, \dfrac{4}{3}\sin t, t\right\rangle;\quad t=0$

73.* Suppose that $f(x)$ is at least twice differentiable on an interval containing a, with nonzero first and second derivatives at $x=a$. Let $C(c_1,c_2)$ be the center of the osculating circle of the plane curve $y=f(x)$ at $(a,f(a))$. Prove that c_1 and c_2 can be determined as follows.

$$c_1=a-\frac{f'(a)\left(1+\left[f'(a)\right]^2\right)}{f''(a)}$$

$$c_2=f(a)+\frac{1+\left[f'(a)\right]^2}{f''(a)}$$

74–75 The center of the osculating circle of the curve $\mathbf{r}(t)$ at P is called the curve's **center of curvature** at P, while the radius of the osculating circle is the **radius of curvature** of $\mathbf{r}(t)$ at P. The locus of all centers of curvature is called the **evolute** of $\mathbf{r}(t)$. Exercises 74 and 75 will use this concept.

74. Show that the evolute of the parabola $y=x^2$ can be parametrized as $\mathbf{r}(t)=\left\langle -4t^3, 3t^2+\frac{1}{2}\right\rangle$. (**Hint:** Use Exercise 73.)

75. Find the evolute for the curve $y=x^3$ $(x>0)$.

76–87 Find the tangential and normal components of acceleration for the given position function.

76. $\mathbf{r}(t)=\langle \sin t, \cos t, 2t\rangle$

77. $\mathbf{r}(t)=\langle t, 2\sin t, 2\cos t\rangle$

78. $\mathbf{r}(t)=\langle 2t, t^2, 0\rangle$

79. $\mathbf{r}(t)=\langle 2t, t^2, t\rangle$

80. $\mathbf{r}(t)=\langle 2t, t^2, \ln t\rangle$

81. $\mathbf{r}(t)=\langle t-\sin t, 1-\cos t, 0\rangle$

82. $\mathbf{r}(t)=\left\langle \dfrac{1}{t}, \sqrt{2}t, \dfrac{1}{3}t^3\right\rangle$

83. $\mathbf{r}(t)=\left\langle 2t, t^2, \dfrac{t^3}{3}\right\rangle$

84. $\mathbf{r}(t)=\left\langle \dfrac{\sqrt{2}}{2}t, \dfrac{\sqrt{2}}{2}t, -\ln(\cos t)\right\rangle$

85. $\mathbf{r}(t)=\langle e^t, e^{-t}, \sqrt{2}t\rangle$

86.* $\mathbf{r}(t)=\langle bt, a\cos\omega t, a\sin\omega t\rangle$

87.* $\mathbf{r}(t)=\langle e^t, e^t\cos t, e^t\sin t\rangle$

88. According to Newton's Second Law of Motion, the magnitude of the friction force that keeps a car from skidding out of a curve is $\mathbf{F}(t) = ma_N(t)$, where m is the mass of the car, and $a_N(t)$ is the normal component of acceleration. Find the minimum friction force needed to keep a 1500 kg car from skidding as it navigates a curve of radius 15 m, at a constant speed of 43 km/h. (Note that such a normal force, also called centripetal or center-seeking force, is needed to keep the car on a circular path.)

89. Generalize your solution to Exercise 88 to obtain a formula for the centripetal force acting on an object of mass m that moves along a circular path of radius r at a constant speed of v.

90. Suppose an object is moving along the space curve $\mathbf{r}(t)$ when at time $t = t_0$ its velocity vector is $\mathbf{v}(t_0) = \langle 32, -10, 15 \rangle$ and its acceleration is $\mathbf{a}(t_0) = \langle 2, 1, -3 \rangle$. Is the object's speed increasing or decreasing at this instant?

91. When navigating a curve, a driver should minimize the normal component of acceleration, which, if too big, could cause the car to skid. According to common advice, a driver should slow down before entering a curve, and then gently accelerate once in the curve. Use your knowledge of the acceleration vector to explain why this is sound advice. (Mention changes in the tangential and normal components of the acceleration vector.)

92.* The radius of the "Singapore Flyer," which was the world's tallest Ferris wheel from 2008 to 2014, is 75 meters. Suppose the wheel is rotating at an angular speed of 0.02 radians per second, which is increasing at a rate of 1.33×10^{-4} radians per second. Find the tangential and normal components, as well as the magnitude of acceleration of the riders, in a capsule that is at the very top of the wheel at this instant.

93. Prove the second Frenet-Serret formula,

$$\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}.$$

(**Hint:** Noting that $d\mathbf{T}/ds = \kappa \mathbf{N}$ and $d\mathbf{B}/ds = -\tau \mathbf{N}$, use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.)

94. Prove that the tangential component of acceleration can be computed by the formula $a_T = (\mathbf{r}' \cdot \mathbf{r}'')/|\mathbf{r}'|$. (**Hint:** Making a sketch is helpful.)

95. Prove that $a_N = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}$. (See the hint given in Exercise 94.)

96. Prove the following formula.

$$\mathbf{r}''' = \left[s''' - \kappa^2 (s')^3\right]\mathbf{T} + \left[3\kappa s' s'' + \kappa' (s')^2\right]\mathbf{N} + \kappa\tau (s')^3 \mathbf{B}$$

97. Prove the formula $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$.

(**Hint:** Use Exercise 96.)

Concept Check

98–106 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

98. $\mathbf{T} \times \mathbf{B} = \mathbf{0}$

99. $(\mathbf{T} \times \mathbf{N}) \cdot \mathbf{B} = 0$

100. $\left|\dfrac{d(\mathbf{T} \cdot \mathbf{T})}{ds}\right| = \kappa^2$

101. The radius of the osculating circle for $y = \sin x$ at the origin is 1.

102. The acceleration vector of a particle moving on a curve $\mathbf{r}(t)$ is always in the osculating plane.

103. If we double the speed of a car in a curve, the force required to keep it from skidding is also doubled.

104. If a car is moving in a curve, then its acceleration is perpendicular to the direction of motion.

105. If an object is moving along a smooth curve that is not a straight line, then a normal force is acting on the object.

106. If the acceleration of a moving object is nonzero, and not a multiple of $\mathbf{N}$, then its speed is changing.

12.3 Technology Exercises

107–109 Use a graphing utility to plot the curves in Exercises 79, 83, and 87. Then find and graph their respective curvature and torsion functions. Interpret these graphs in terms of your three-dimensional plot.

107. $\mathbf{r}(t) = \langle 2t, t^2, t \rangle$

108. $\mathbf{r}(t) = \left\langle 2t, t^2, \dfrac{t^3}{3} \right\rangle$

109. $\mathbf{r}(t) = \langle e^t, e^t \cos t, e^t \sin t \rangle$

110. Write a program for a computer algebra system or programmable calculator that returns the parametric form of the osculating circle of a given space curve at a specified point. Use the program to check your answers for Exercises 69–72.