12.2 Exercises

1–10 Find the arc length of the curve over the given interval.

1. $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 2t \rangle; \quad [0, 6\pi]$
2. $\mathbf{r}(t) = \langle t, 2\sin t, 2\cos t \rangle; \quad [0, 4\pi]$
3. $\mathbf{r}(t) = \langle 2-5t, 1+3t, 4-\sqrt{2}t \rangle; \quad [0, 3]$
4. $\mathbf{r}(t) = \langle e^t, 2e^t \cos t, 2e^t \sin t \rangle; \quad [0, 1]$
5. $\mathbf{r}(t) = \langle 2t, t^2, \ln t \rangle; \quad [1, e]$
6. $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, t^2, 2t \right\rangle; \quad [0, 2]$
7. $\mathbf{r}(t) = \langle t^3, \sqrt{6}t^2, 4t \rangle; \quad [0, 5]$
8. $\mathbf{r}(t) = \langle t^2, 2t\sin t, 2t\cos t \rangle; \quad [0, 1]$
9. $\mathbf{r}(t) = \left\langle \frac{\sqrt{2}}{2}t, \frac{\sqrt{2}}{2}t, -\ln(\cos t) \right\rangle; \quad \left[0, \frac{\pi}{3}\right]$
10. $\mathbf{r}(t) = \langle t - \tanh t, \operatorname{sech} t \rangle; \quad [0, \ln 3]$ (This curve is called a *tractrix*.)

11–16 Reparametrize the given curve with respect to arc length.

11. The line $\mathbf{r}(t) = \langle 1+2t, 3-5t, 4+4t \rangle$
12. The circle $\mathbf{r}(t) = \langle 0, 4\cos t, 4\sin t \rangle$
13. The helix $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$
14. The curve $\mathbf{r}(t) = \langle t, \cosh t, \sinh t \rangle$
15. The helix $\mathbf{r}(t) = \langle bt, a\cos \omega t, a\sin \omega t \rangle$
16. The curve $\mathbf{r}(t) = \langle e^t, e^t \cos t, e^t \sin t \rangle$

17–24 Find the unit tangent vector for the given curve.

17. $\mathbf{r}(t) = \langle t+1, t^3, -t^2 \rangle$
18. $\mathbf{r}(t) = \langle 2t, \cos t, \sin t \rangle$
19. $\mathbf{r}(t) = \langle e^t, e^t \cos t, e^t \sin t \rangle$
20. $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, \frac{1}{t}, \sqrt{2}t \right\rangle$
21. $\mathbf{r}(t) = \langle t\cos t - \sin t, t, t\sin t + \cos t \rangle$
22. $\mathbf{r}(t) = \langle bt, a\cos \omega t, a\sin \omega t \rangle$
23. $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t \rangle$
24. $\mathbf{r}(t) = \langle t, \sqrt{2-2t^2}, t \rangle$

25. Find an arc length parametrization of the straight line $y = mx + b$.

26. A circle of radius 5 is located in the plane $x = 3$, centered at $(3, 2, 1)$. Find an arc length parametrization for this circle.

27. The following are all parametrizations of the same helix. Which one is the arc length parametrization?

 a. $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{t}{3} \right\rangle$

 b. $\mathbf{r}(t) = \langle \cos 3t, \sin 3t, t \rangle$

 c. $\mathbf{r}(t) = \left\langle \cos \frac{3t}{\sqrt{10}}, \sin \frac{3t}{\sqrt{10}}, \frac{t}{\sqrt{10}} \right\rangle$

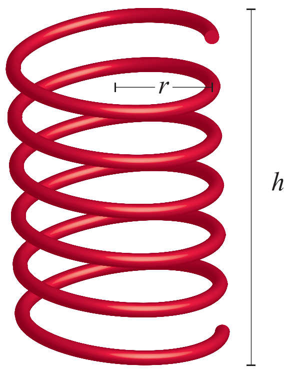
 d. $\mathbf{r}(t) = \left\langle \cos \sqrt{10}t, \sin \sqrt{10}t, \frac{\sqrt{10}t}{3} \right\rangle$

28. Suppose a bug starts crawling at $(0, 0, 0)$ along the curve $\mathbf{r}(t) = \langle 3t, 2t^2, 4\sqrt{2/3}\,t^{3/2} \rangle$. After crawling exactly 5 unit lengths, it runs into a spider web. Find the coordinates of the point where the curve pierces the spider web.

29. Recall from Section 9.1 the parametrization of the first full arch of the cycloid: $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, $\theta \in [0, 2\pi]$. Find the arc length parametrization of this curve.

30. One version of the Bernoulli spiral can be parametrized as $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$ (see Exercise 85 of Section 12.1). Find the arc length parametrization of this curve. (**Hint:** For the lower limit of integration in calculating $s(t)$, use $-\infty$.)

31.* Suppose a spring has radius r and it reaches height h while making n full revolutions. Find a formula for the length of the wire used in manufacturing this spring.



32. Suppose you calculate the arc length parametrization for the curve $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$, while your classmate does the same, but she starts out with the parametrization $\mathbf{r}(t)=\langle f(t^3),g(t^3),h(t^3)\rangle$. Do you obtain equivalent answers? Explain.

33. Use the vector function given in Example 5, $\mathbf{r}(t)=\langle(v_0\cos\theta)t+x_0,-\frac{1}{2}gt^2+(v_0\sin\theta)t+y_0\rangle$, to find a formula for the range of the projectile, assuming it was launched at ground level. (The range of the projectile is the distance between its launching and landing points.)

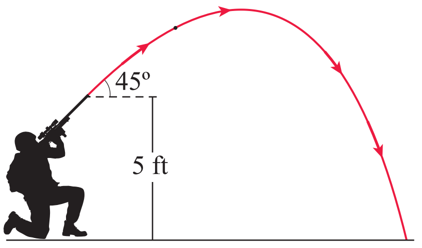
34. Use the formulas found in Example 5 and Exercise 33 to revisit Exercise 81 of Section 12.1.

35. Use Exercise 33 to find the angle θ which corresponds to the maximum range for the projectile.

36.* Repeat Exercise 35 to find the angle θ that corresponds to the maximum downhill range if the projectile is launched on a downhill terrain that drops at an angle of δ from horizontal. (θ is still the angle of elevation measured from horizontal.)

37.* Repeat Exercise 36 to find the angle θ that corresponds to the maximum uphill range if the projectile is launched on an uphill terrain with an angle of elevation of φ from horizontal.

38. A pellet is shot from an air rifle with a muzzle velocity of 1200 ft/s, leaving the rifle 5 ft above ground level and at a 45° angle of elevation. Assuming the surrounding terrain is flat and level, how far does the pellet travel, and with what speed does it hit the ground? (As usual, ignore air resistance.)



39. A projectile is launched from a 2 m high platform with an initial speed of 30 m/s, in a direction 60° upward from horizontal. Ignore air resistance.

a. Find a vector function that models the projectile's path.

b. Find the maximum height attained by the projectile, its range, and the speed of impact. (**Hint:** Suppose the projectile was launched from the point $(0,2)$.)

40. Answer the questions of Exercise 39, assuming that the launch took place on the moon. (Use $g/6$ for the acceleration caused by gravity near the moon's surface.)

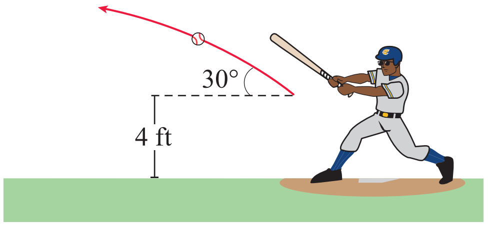
41. The exit velocity of a baseball (its velocity as it leaves the bat) is 128 feet per second, in the direction of 30° above horizontal. If it was hit 4 feet above ground level, find

a. a vector function that models the path of the baseball,

b. the maximum height attained by the baseball,

c. the horizontal distance traveled by the ball and its speed of impact.

(Ignore air resistance.)



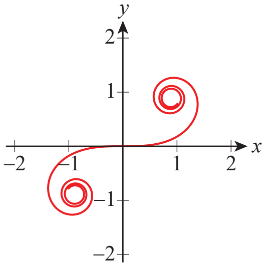
42. A golf ball is hit with an initial speed of 140 feet per second, 50° upward from horizontal, toward a hole 158 yards away (when measured horizontally). If the elevation of the hole is 121 feet higher than that of the starting point of the golf ball, will the ball land in the hole? (**Hint:** Use a vector function to examine the trajectory of the golf ball. Ignore air resistance.)

43.* A projectile is launched from a 6-foot platform with an initial speed of 200 feet per second and at a firing angle of x degrees above horizontal. Find the value of x that will result in a range of 1000 feet. (Ignore all retarding forces but gravity. Express your answer in degrees.)

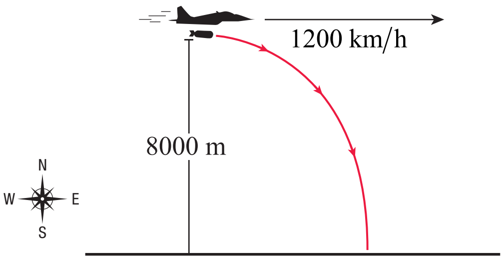
44. The curve pictured below is a *Cornu spiral* over the interval $[-2\pi,2\pi]$ (also known as *Euler's spiral*, though initially discovered by Johann Bernoulli). It is defined by

$$\mathbf{r}(t)=\left\langle\int_0^t\cos\frac{u^2}{2}\,du,\int_0^t\sin\frac{u^2}{2}\,du\right\rangle.$$

Find the arc length of the Cornu spiral over the interval $[0,b]$.



45. A bomber plane is flying eastward at a speed of 1200 kilometers per hour when it releases a bomb at an altitude of 8000 meters. Ignoring air resistance, find a vector function modeling the path of the bomb, the horizontal distance traveled by the bomb, and its speed of impact. (**Hint:** Suppose the bomb was released at the point $(0,8000)$, and that the x-axis points to the east.)

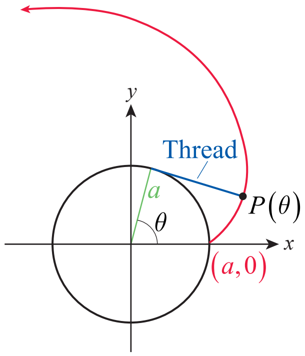


46. Suppose a certain type of medieval cannon, at a 10-degree angle of elevation, is able to fire to a distance of 2400 feet. Find the initial speed of the cannonball. What maximum range can be achieved with this type of cannon? (**Hint:** To simplify matters, suppose that the cannon ball was shot from ground level. Ignore air resistance. See Exercises 33 and 35.)

47.* When air resistance is taken into consideration, assuming it is proportional to the projectile's velocity, then the vector function $\mathbf{r}(t)$ for the projectile's motion (see Example 5) satisfies the differential equation $\frac{d^2}{dt^2}\mathbf{r}(t) = -C\frac{d}{dt}\mathbf{r}(t) - g\,\mathbf{j}$, where C is the drag coefficient. Assuming that the projectile is launched from the origin (i.e., $x_0 = y_0 = 0$), find $\mathbf{r}(t)$ under these conditions.

48. Suppose that the path of a moving point is a straight line. Prove that in this case, $\mathbf{T}'(t) = \mathbf{0}$.

49.* If we unwind a thread from a fixed circular spool of radius a, starting at the point $(a,0)$ and keeping the thread taut in the xy-plane throughout the process, the curve traced out by the endpoint of the thread is called the *involute* of the circle. (**Note:** In the figure below, the point $P(\theta)$ denotes the endpoint of the thread at the instant when the radius to the point of tangency makes an angle of θ with the positive x-axis.)



a. Use the figure to derive the following parametrization of the involute of a circle of radius a.

$$\mathbf{r}(\theta) = a\langle \cos\theta + \theta\sin\theta, \sin\theta - \theta\cos\theta \rangle,\ \theta > 0$$

b. Reparametrize the involute with respect to arc length.

Concept Check

50–54 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

50. The circle $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$ has the property that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular for all t.

51. The helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ has the property that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular for all t.

52. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and $f(t)$, $g(t)$, and $h(t)$ are linear polynomials, then $\mathbf{T}(t)$ is constant.

53. When the launching speed of a projectile is doubled, its range doubles. (Suppose it is launched at angle α, $0 < \alpha < \pi/2$, upward from horizontal.)

54. For a space curve $\mathbf{r}(t)$, we have $|\mathbf{r}(t)|' = |\mathbf{r}'(t)|$.