Example 7 Using Integration to Find Velocity and Position Vector Functions

A ball is shot by a slingshot into the air with an initial velocity vector $\mathbf{v}(0)=\langle 0,10,64\rangle$, measured in ft/s. Determine its velocity $\mathbf{v}$ and its position $\mathbf{r}$ as functions of time t. Find the positions of the ball at 2 seconds and 4 seconds.

Solution

We begin with the fact that the acceleration of the ball is given by the vector $\mathbf{a}(t)=\langle 0,0,-32\rangle$, reflecting the fact that Earth's gravity is pulling it (and every other object) in the negative z-direction at a rate of $32\ \text{ft/s}^2$. So the ball's velocity vector is the indefinite integral of $\mathbf{a}$.

$$\mathbf{v}(t)=\int \mathbf{a}(t)\,dt=\langle 0,0,-32t\rangle+\mathbf{C}$$

Since we are given its velocity at time $t=0$,

$$\langle 0,10,64\rangle=\mathbf{v}(0)=\langle 0,0,-32\cdot 0\rangle+\mathbf{C} \quad\Rightarrow\quad \mathbf{C}=\langle 0,10,64\rangle,$$

so $\mathbf{v}(t)=\langle 0,10,-32t+64\rangle$.

If we let $\mathbf{r}$ denote the ball's position vector, then

$$\mathbf{r}(t)=\int \mathbf{v}(t)\,dt=\langle 0,10t,-16t^2+64t\rangle+\mathbf{C}.$$

We can locate the origin of the coordinate system wherever convenient, so we may as well specify that the ball's position at time $t=0$ is the origin, meaning $\mathbf{C}=\langle 0,0,0\rangle$. We previously determined that the ball reaches its maximum height when $t=2$ and lands on the ground when $t=4$, and its position at these times is given by the following definite integrals of its velocity.

$$\text{Position at } t=2\text{:}\ \int_0^2 \mathbf{v}(t)\,dt=\langle 0,20,64\rangle$$

$$\text{Position at } t=4\text{:}\ \int_0^4 \mathbf{v}(t)\,dt=\langle 0,40,0\rangle$$

In words, the ball has traveled 20 feet horizontally when it reaches its maximum height of 64 feet, and has traveled 40 feet horizontally when it lands.

12.1 Exercises

1–4 Find the domain of the vector function. If possible, evaluate the vector function at the indicated points.

1. $\mathbf{r}(t)=\frac{1}{t^2}\mathbf{i}+2t\,\mathbf{j}-t\,\mathbf{k}$; **a.** $t=2$ **b.** $t=-5$

2. $\mathbf{r}(t)=3t\,\mathbf{i}-e^t\,\mathbf{j}-\sqrt{t-1}\,\mathbf{k}$; **a.** $t=2$ **b.** $t=-5$

3. $\mathbf{r}(t)=\frac{1}{\sqrt{9-t^2}}\mathbf{i}-t^3\,\mathbf{j}+\ln t\,\mathbf{k}$; **a.** $t=-4$ **b.** $t=1$

4. $\mathbf{r}(t)-\mathbf{s}(t)$, where $\mathbf{r}(t)=\sqrt{t}\,\mathbf{i}-5t^2\,\mathbf{k}$, $s(t)=e^{-t}\,\mathbf{i}+t^2\,\mathbf{j}$; **a.** $t=-1$ **b.** $t=1$

5. If $\mathbf{r}(t)=\mathbf{i}-t\,\mathbf{j}$ and $\mathbf{s}(t)=t\,\mathbf{i}+3\,\mathbf{j}+t^3\,\mathbf{k}$, find a formula for $\mathbf{u}(t)=\mathbf{r}(t)\cdot\mathbf{s}(t)$. Is it a space curve?

6. Repeat Exercise 5 for $\mathbf{v}(t)=\mathbf{r}(t)\times\mathbf{s}(t)$.

7–14 Match the vector function with its graph (labeled A–H).

7. $\mathbf{r}(t)=\left\langle \frac{1}{2}t\cos t, \frac{1}{2}t\sin t, \frac{t}{2} \right\rangle;\quad t\in[0,6\pi]$

8. $\mathbf{r}(t)=\left\langle 5\sin^2 t, \cos^2 t, t \right\rangle;\quad t\in[0,4\pi]$

9. $\mathbf{r}(t)=\left\langle \ln t, \sin t, \cos t \right\rangle;\quad t\in(0,6\pi)$

10. $\mathbf{r}(t)=\left\langle 3\cos(t^2), 3\sin(t^2), 3\sqrt{t} \right\rangle;\quad t\in[0,4\pi]$

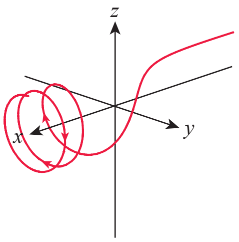
11. $\mathbf{r}(t)=\left\langle \cos\sqrt{t}, \sin\sqrt{t}, \frac{t}{50} \right\rangle;\quad t\in[0,36\pi^2]$

12. $\mathbf{r}(t)=\left\langle 2\sin t, 4\cos t, \frac{t}{4} \right\rangle;\quad t\in[0,6\pi]$

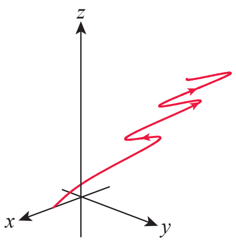
13. $\mathbf{r}(t)=\left\langle 1+t^2, 2t, 2t \right\rangle;\quad t\in[-6,25]$

14. $\mathbf{r}(t)=\left\langle \cos t, \sqrt{t}, \sqrt{t} \right\rangle;\quad t\in[0,6\pi]$

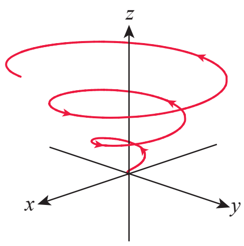
A.



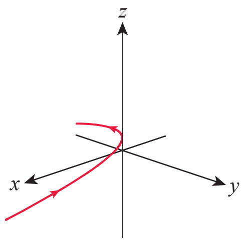
B.



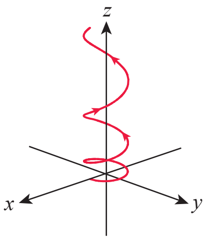
C.



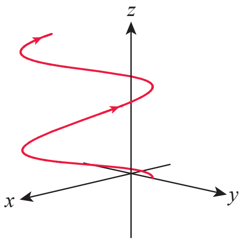
D.



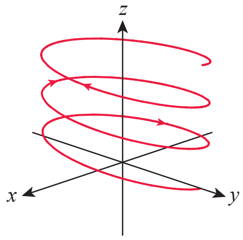
E.



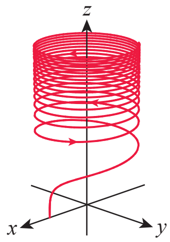
F.



G.



H.



15–22 Sketch the space curve by hand. (**Hint:** See Exercises 7–14.)

15. $\mathbf{r}(t)=\left\langle \sin t, \cos t, \sqrt{t} \right\rangle;\quad t\in[0,4\pi]$

16. $\mathbf{r}(t)=\left\langle 2t\cos t, 3t\sin t, t \right\rangle;\quad t\in[0,6\pi]$

17. $\mathbf{r}(t)=\left\langle t, t, 3\sin t \right\rangle;\quad t\in[-2\pi,2\pi]$

18. $\mathbf{r}(t)=\left\langle 2\cos(t^2), 4\sin(t^2), 2\sqrt{t} \right\rangle;\quad t\in\left[0,2\sqrt{3\pi}\right]$

19. $\mathbf{r}(t)=\left\langle \cos t, \sin t, \ln t \right\rangle;\quad t\in\left[\frac{1}{e^3},6\pi\right]$

20. $\mathbf{r}(t)=\left\langle \frac{t}{3}, \frac{3\cos 2t}{t}, \frac{3\sin 2t}{t} \right\rangle;\quad t\in(0,4\pi)$

21. $\mathbf{r}(t)=\left\langle \frac{t}{2}, 4\sin t, 2\cos t \right\rangle;\quad t\in[0,4\pi]$

22. $\mathbf{r}(t)=\left\langle 4\cos t, 4\sin t, \cos 10t \right\rangle;\quad t\in[0,2\pi]$

23–29 Describe the intersection of the surfaces as a vector function. (Use the suggested parameter.)

23. The elliptic cylinder $2x^2+3y^2=6$ and the plane $3x+2z=2$ $(x=\sqrt{3}\cos t)$

24. The paraboloid $x^2+y^2=z$ and the plane $x+z=2$ $(z=t)$

25. The cylinder $x^2+y^2=1$ and the hyperbolic paraboloid $2x^2-y^2=z$ $(x=\cos t)$

26. The cylinder $x^2+y^2=9$ and the surface $y=z/x$ $(x=3\cos t)$

27. The elliptic paraboloid $2x^2+y^2=2z$ and the parabolic cylinder $y^2=x$ $(y=t)$

28. The cone $x^2+y^2=z^2$ and the plane $2z=y+4$ $(y=t)$

29. The semiellipsoid $\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{8}=1,\ z\ge 0$ and the parabolic cylinder $2x=y^2$ $(y=t)$

30–35 Determine whether the indicated limit exists. If so, find it.

30. $\lim_{t\to -1}\left\langle t^2-2t,\sqrt{t+5},\frac{1}{t}\right\rangle$

31. $\lim_{t\to 1}\left\langle e^{-3t},\sqrt{t-2},\cos 2t\right\rangle$

32. $\lim_{t\to 3}\left\langle \frac{t+2}{t-3},\ln t,\cot \pi t\right\rangle$

33. $\lim_{t\to 0}\left\langle \ln\left(t^2+1\right),|t|,2^t\right\rangle$

34. $\lim_{t\to 0}\left\langle \sqrt{1-\cos t},e^{\tan t},\frac{2\sin t}{t}\right\rangle$

35. $\lim_{t\to 0}\left\langle \frac{2t}{t+1},\ln\left(t+1\right),\sin\frac{\pi}{t}\right\rangle$

36–39 Find any discontinuities of the given vector function.

36. $\mathbf{r}(t)=e^t\,\mathbf{i}+\frac{2}{t+1}\mathbf{j}+3t^2\mathbf{k}$

37. $\mathbf{r}(t)=2t^2\,\mathbf{i}-5|t|\,\mathbf{j}+\cos t\,\mathbf{k}$

38. $\mathbf{r}(t)=\mathbf{i}+\frac{t}{t^2+1}\mathbf{j}-\cot t\,\mathbf{k}$

39. $\mathbf{r}(t)=\left(t^3-1\right)\mathbf{i}-\sin\frac{\pi}{t}\mathbf{j}-\sqrt{t^2+2}\,\mathbf{k}$

40. Prove: If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector functions so that both have limits at $t=t_0$, then the limit of their dot product is the dot product of their limits, that is,

$$\lim_{t\to t_0}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]=\lim_{t\to t_0}\mathbf{u}(t)\cdot\lim_{t\to t_0}\mathbf{v}(t).$$

41. Prove: If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector functions so that both have limits at $t=t_0$, then the limit of their cross product is the cross product of their limits, that is,

$$\lim_{t\to t_0}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\lim_{t\to t_0}\mathbf{u}(t)\times\lim_{t\to t_0}\mathbf{v}(t).$$

(**Hint:** Use the determinant rule for determining cross products.)

42. Prove that the differentiability of a vector function implies its continuity, that is, if $\mathbf{u}(t)$ is differentiable at $t=t_0$, then $\mathbf{u}(t)$ is continuous at $t=t_0$.

43. Give a rigorous proof of the fact that the vector function $\mathbf{r}(t)=\left\langle f(t),g(t),h(t)\right\rangle$ is continuous if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous.

44. Prove that if the vector function $\mathbf{r}(t)$ is continuous at $t=t_0$, then the scalar function $|\mathbf{r}(t)|$ is also continuous at $t=t_0$.

45. Prove that the converse of Exercise 44 is false by finding a vector function $\mathbf{r}(t)$ with a discontinuity at $t=t_0$ so that $|\mathbf{r}(t)|$ is continuous at $t=t_0$.

46–51 Find $\mathbf{r}'(t)$.

46. $\mathbf{r}(t)=t^2\,\mathbf{i}+2\sqrt{t}\,\mathbf{j}-t\mathbf{k}$

47. $\mathbf{r}(t)=\left(2-t^3\right)\mathbf{i}-\pi^2\,\mathbf{j}+\frac{t^5}{5}\mathbf{k}$

48. $\mathbf{r}(t)=\sin t\,\mathbf{i}+e^t\,\mathbf{j}-\tan t\,\mathbf{k}$

49. $\mathbf{r}(t)=\ln t\,\mathbf{i}-\csc t\,\mathbf{j}+\sqrt{4-t^2}\,\mathbf{k}$

50. $\mathbf{r}(t)=\left\langle \frac{1}{\sqrt[3]{t^2}},\arctan 2t,\sin^3 t\right\rangle$

51. $\mathbf{r}(t)=\left\langle \cos\left(t^2+1\right),\frac{t+1}{t-1},3\arcsin t\right\rangle$

52–55 Find a unit vector that is tangent to the graph of the vector function at the specified value of *t*.

52. $\mathbf{r}(t)=t\,\mathbf{i}-2t^2\,\mathbf{j}+2\sqrt{t}\,\mathbf{k};\quad t=1$

53. $\mathbf{r}(t)=2\sin t\,\mathbf{i}-e^t\,\mathbf{j}+8\sqrt{4+t}\,\mathbf{k};\quad t=0$

54. $\mathbf{s}(t)=\left\langle \arctan t,-\cos^2 t,\sqrt{3}t\right\rangle;\quad t=0$

55. $\mathbf{u}(t)=\left\langle 4t,3\sin t,3\cos t\right\rangle;\quad t=0$

56–59 Find the vector form of an equation for the line tangent to the curve at the specified value of *t*.

56. $\mathbf{r}(t)=2t\,\mathbf{i}+\left(t^2-4\right)\mathbf{j}+\sqrt{t+1}\,\mathbf{k};\quad t=3$

57. $\mathbf{r}(t)=e^{3t}\mathbf{i}-e^{2t}\mathbf{j}+e^t\,\mathbf{k};\quad t=0$

58. $\mathbf{s}(t)=\left\langle t,\sin 2t,\cos 2t\right\rangle;\quad t=0$

59. $\mathbf{u}(t)=\left\langle \arcsin t,\arccos t,\ln t\right\rangle;\quad t=\frac{1}{2}$

60–64 Prove the indicated differentiation rule, assuming that **u** and **v** are differentiable vector functions, **C** is a constant vector, *c* is a scalar, and *f* is a differentiable scalar function.

60. Constant Vector Rule: $\frac{d}{dt}\mathbf{C} = \mathbf{0}$

61. Scalar Multiple Rules: $\frac{d}{dt}\left[c\,\mathbf{u}(t)\right] = c\,\mathbf{u}'(t)$ and $\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

62. Sum/Difference Rules: $\frac{d}{dt}\left[\mathbf{u}(t) \pm \mathbf{v}(t)\right] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$

63. Dot Product Rule: $\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

64. Chain Rule: $\frac{d}{dt}\left[\mathbf{u}(f(t))\right] = f'(t)\mathbf{u}'(f(t))$

65. Prove the following differentiation rule for the triple scalar product of vector functions: If $\mathbf{u}(t)$, $\mathbf{v}(t)$, and $\mathbf{w}(t)$ are differentiable, then

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}.$$

66. Prove that if a point moves along a sphere, then its velocity vector is tangential to the sphere. (**Hint:** See Example 6.)

67. Prove that if a point moves along a curve in $\mathbb{R}^3$ with constant speed, then its velocity and acceleration vectors are orthogonal. (**Hint:** See Exercise 66.)

68. (A converse of Example 6) Assume $\mathbf{r}(t)$ is a differentiable vector function satisfying $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$ for all *t*. Show that $|\mathbf{r}(t)|$ is constant, that is, the graph of $\mathbf{r}(t)$ lies on a sphere centered at the origin.

69–74 Find the indefinite integral.

69. $\int\left\langle 3t^2, t^3 - t, -\sqrt{t}\right\rangle dt$

70. $\int\left(\frac{t}{t^2+1}\mathbf{i} - \frac{1}{t^2+1}\mathbf{j} + \frac{t}{t^2-1}\mathbf{k}\right)dt$

71. $\int\left(t\,\mathbf{i} + 3\,\mathbf{j} - 4t^3\,\mathbf{k}\right)dt$

72. $\int\left(\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - \sec^2 t\,\mathbf{k}\right)dt$

73. $\int\left(2\,\mathbf{i} - \frac{1}{t}\mathbf{j} + t^{3/2}\,\mathbf{k}\right)dt$

74. $\int\left\langle \frac{1}{t^2}, \ln t, -e^{-t}\right\rangle dt$

75–80 Evaluate the definite integral.

75. $\int_0^3\left[(2-t)\mathbf{i} - 4\,\mathbf{j} + t^2\,\mathbf{k}\right]dt$

76. $\int_0^1\left[2t^4\,\mathbf{i} + t\,\mathbf{j} - \left(t^2 - 2\right)\mathbf{k}\right]dt$

77. $\int_{-1}^1\left\langle \sqrt[3]{t}, t, t^4\right\rangle dt$

78. $\int_0^\pi\left(\sin t\,\mathbf{i} + t\sin t\,\mathbf{j} - \mathbf{k}\right)dt$

79. $\int_1^e\left\langle 2e^t, -\ln t, \frac{1}{t}\right\rangle dt$

80. $\int_0^3\left\langle \sqrt{t+1}, \frac{6t}{t^2+1}, \frac{-1}{(t+1)(t-4)}\right\rangle dt$

81. A projectile is launched from the ground with an initial speed of 78.48 m/s at an angle of elevation of 30° from horizontal. After determining the vector functions $\mathbf{v}(t)$ and $\mathbf{r}(t)$, as in Example 7, find the maximum altitude reached by the projectile as well as its range. (Suppose that the launch takes place in the positive *x*-direction. Use $g \approx 9.81\text{ m/s}^2$ and ignore air resistance.)

82. Use Exercise 81 to determine the effect on the maximum altitude and range of the projectile if we double its initial velocity.

83. A particle is moving in $\mathbb{R}^3$ so that its acceleration function is $\mathbf{a}(t) = \langle 2t, 1, 0\rangle$. Find the velocity and position functions of the particle if it starts at the point $\mathbf{r}(0) = \langle -5, 0, 2\rangle$ with initial velocity $\mathbf{v}(0) = \langle 3, 1, -1\rangle$.

84.* Prove that the force acting on a mass moving along a circle of radius *R* with constant angular speed ω is always pointing toward the center of the circle. (Such a force is called a *center-seeking* or *centripetal* force.) (**Hint:** Parametrize the path of the object and differentiate twice to find its acceleration, then use Newton's Second Law of Motion.)

85.* The plane curve $\mathbf{r}(t) = \left\langle ae^{bt}\cos t, ae^{bt}\sin t\right\rangle$ is called a *logarithmic spiral* or *Bernoulli spiral*. One of its intriguing properties is that for any fixed point $P = \mathbf{r}(t_0)$, the corresponding radial and tangent lines form a constant angle φ. Prove this fact, and find the angle φ.

86.* The *angular momentum* (with respect to the origin) of a mass m that is moving along a space curve $\mathbf{r}(t)$, is defined as

$$\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$$

Use Newton's Second Law to demonstrate that $\boldsymbol{\tau}$, the net external torque acting on m, is equal to the derivative of $\mathbf{L}(t)$; that is,

$$\boldsymbol{\tau} = \mathbf{L}'(t).$$

(**Hint:** Use the fact that torque is the cross product of the displacement vector and the force vector, that is, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.)

12.1 Technology Exercises

87–92. Use the integration capabilities of a graphing utility to verify your answers to Exercises 69–74.

93–98. Use the integration capabilities of a graphing utility to verify your answers to Exercises 75–80.

99–104. Use the `Limit` command of a graphing utility to verify your answers to Exercises 30–35.

105–112. Use a graphing utility to graph the curves of Exercises 15–22.

113–119. Use a graphing utility to display the curves of Exercises 23–29 as intersections of the given surfaces.