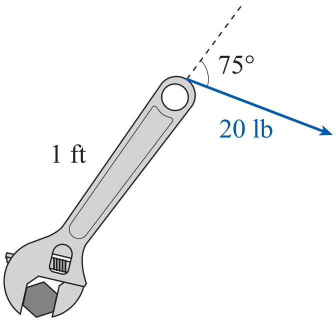

Figure 9

Example 7 Using a Cross Product to Find the Torque Generated by a Wrench

A bolt is tightened by a wrench supplying 20 pounds of force at an angle of $\theta = 75°$ (see Figure 9). The length of the wrench is 1 foot. How much torque is applied to the bolt at the pivot point?

Solution

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta = (1)(20)\sin 75° \approx 19.32 \text{ lb-ft}$$

Note that the direction of the torque vector is into the page.

11.4 Exercises

1–6 Use the determinant formula to find the cross product.

1. $\langle 2,-5,1\rangle \times \langle 1,1,-1\rangle$ **2.** $\langle -3,0,1\rangle \times \langle 2,1,4\rangle$ **3.** $\langle 5,-5,2\rangle \times \langle 3,1,-1\rangle$

4. $\left\langle \frac{1}{2},2,-3\right\rangle \times \left\langle 1,\frac{3}{2},-1\right\rangle$ **5.** $\left\langle \frac{1}{3},3,0\right\rangle \times \left\langle 0,-\frac{5}{3},-\frac{1}{4}\right\rangle$ **6.** $\langle 0.2,-0.8,1.25\rangle \times \langle -2,8,-12.5\rangle$

7–13 Use the determinant formula to prove the indicated property of the cross product. (Assume **u**, **v**, and **w** represent vectors in $\mathbb{R}^3$, while *a* and *b* represent scalars.)

7. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ **8.** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ **9.** $(a\mathbf{u}) \times (b\mathbf{v}) = (ab)(\mathbf{u} \times \mathbf{v})$

10. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ **11.** $\mathbf{0} \times \mathbf{u} = \mathbf{0}$ **12.** $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

13. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

14. Prove that the cross products of the standard unit vectors obey the following vector equations.

a. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ **b.** $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ **c.** $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

15–18 Notice that using the properties of the cross product and results of Exercise 14, we can evaluate cross products as illustrated by the following example.

$$\begin{aligned}
(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) &= 2(\mathbf{i} \times \mathbf{i}) - 4(\mathbf{i} \times \mathbf{j}) + 5(\mathbf{i} \times \mathbf{k}) + 6(\mathbf{j} \times \mathbf{i}) - 12(\mathbf{j} \times \mathbf{j}) \\
&\quad + 15(\mathbf{j} \times \mathbf{k}) + 4(\mathbf{k} \times \mathbf{i}) - 8(\mathbf{k} \times \mathbf{j}) + 10(\mathbf{k} \times \mathbf{k}) \\
\text{Note that } \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}. \quad &= -4(\mathbf{i} \times \mathbf{j}) - 5(\mathbf{k} \times \mathbf{i}) - 6(\mathbf{i} \times \mathbf{j}) + 15(\mathbf{j} \times \mathbf{k}) + 4(\mathbf{k} \times \mathbf{i}) + 8(\mathbf{j} \times \mathbf{k}) \\
&= -10(\mathbf{i} \times \mathbf{j}) - (\mathbf{k} \times \mathbf{i}) + 23(\mathbf{j} \times \mathbf{k}) \\
&= -10\mathbf{k} - \mathbf{j} + 23\mathbf{i} \\
&= 23\mathbf{i} - \mathbf{j} - 10\mathbf{k}
\end{aligned}$$

Use the above method to evaluate the cross product $\mathbf{u} \times \mathbf{v}$.

15. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{v} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ **16.** $\mathbf{u} = \frac{1}{3}\mathbf{i} + \frac{5}{2}\mathbf{j} - \mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} - 9\mathbf{j} - 12\mathbf{k}$

17. $\mathbf{u} = \langle 4,-2,1\rangle, \quad \mathbf{v} = \langle 2,5,-6\rangle$ **18.** $\mathbf{u} = \langle -5,1,2\rangle, \quad \mathbf{v} = \langle 3,4,-1\rangle$

19–22 Find both unit vectors perpendicular to **u** and **v**.

19. $\mathbf{u}=\langle 1,0,-1\rangle, \quad \mathbf{v}=\langle 2,-2,1\rangle$

20. $\mathbf{u}=\langle 3,1,0\rangle, \quad \mathbf{v}=\langle -1,0,2\rangle$

21. $\mathbf{u}=\mathbf{j}-3\mathbf{k}, \quad \mathbf{v}=-2\mathbf{i}+\mathbf{j}$

22. $\mathbf{u}=2\mathbf{i}-\mathbf{j}+\mathbf{k}, \quad \mathbf{v}=-\mathbf{i}-\mathbf{j}+3\mathbf{k}$

23–26 Construct a vector normal to the plane containing the indicated points.

23. $P(0,0,0), \quad Q(2,-1,1), \quad R(3,3,4)$

24. $P(-5,0,4), \quad Q(2,2,2), \quad R(6,-1,3)$

25. $P\left(\frac{1}{2},-1,-1\right), \quad Q\left(\frac{5}{2},3,\frac{7}{2}\right), \quad R\left(2,\frac{1}{2},0\right)$

26. $P\left(-\frac{4}{3},-\frac{7}{6},\frac{1}{4}\right), \quad Q\left(-\frac{1}{3},-\frac{5}{6},\frac{3}{4}\right), R\left(\frac{5}{6},-\frac{1}{6},1\right)$

27. Prove that $\overrightarrow{AB}\times\overrightarrow{AC}=\mathbf{0}$ if and only if the points A, B, and C are collinear (i.e., they lie on the same line).

28–31 Use Exercise 27 to check whether the given points are collinear.

28. $P(3,1,1), \quad Q(2,-1,0), \quad R(1,4,-1)$

29. $P(3,1,1), \quad Q(2,-1,0), \quad R(5,5,3)$

30. $P\left(2,0,-\frac{1}{2}\right), \quad Q\left(\frac{1}{2},-1,3\right), \quad R\left(\frac{3}{2},-1,-1\right)$

31. $P\left(0,\frac{1}{3},1\right), \quad Q\left(\frac{1}{3},1,2\right), \quad R\left(-1,-\frac{5}{3},-2\right)$

32–35 Find the area of the parallelogram spanned by the given vectors.

32. $\langle 1,4\rangle, \quad \langle -3,1\rangle$

33. $\langle -3,-2\rangle, \quad \left\langle 1,-\frac{5}{2}\right\rangle$

34. $\langle 7,3\rangle, \quad \langle 6,-10\rangle$

35. $\langle 6,9\rangle, \quad \langle -9,-6\rangle$

36–39 Find the area of the triangle with the given vertices.

36. $A(0,0,0), \quad B(1,2,3), \quad C(-3,-2,-1)$

37. $A(1,1,1), \quad B(4,-2,5), \quad C(-3,1,-1)$

38. $A(-4,1,2), \quad B(-1,3,5), \quad C(-3,0,-5)$

39. $A\left(\frac{1}{2},-1,\frac{5}{2}\right), \quad B\left(\frac{3}{2},-2,\frac{3}{2}\right), \quad C\left(-\frac{3}{2},-3,\frac{7}{2}\right)$

40. Suppose that the vertices of a triangle ABC in the xy-plane have coordinates $(x_A,y_A,0)$, $(x_B,y_B,0)$, and $(x_C,y_C,0)$, respectively. Prove that the area of ΔABC is half the absolute value of the following determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{vmatrix}$$

41–46 Suppose $\mathbf{u}=\langle 1,3,-2\rangle$, $\mathbf{v}=\langle 4,-1,1\rangle$, and $\mathbf{w}=\langle -2,2,-1\rangle$. If possible, evaluate each of the following expressions.

41. $\mathbf{u}+(\mathbf{v}\times\mathbf{w})$

42. $(\mathbf{u}+\mathbf{v})\times\mathbf{w}$

43. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$

44. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

45. $\mathbf{u}\times(\mathbf{v}\cdot\mathbf{w})$

46. $(\mathbf{u}\times\mathbf{v})+\mathbf{w}$

47. Describe the conditions **u** or **v** have to satisfy in order for both their dot product and cross product to be zero, that is, for $\mathbf{u}\cdot\mathbf{v}=\mathbf{0}$ and $\mathbf{u}\times\mathbf{v}=\mathbf{0}$.

48. Prove the following determinant formula for the triple scalar product.

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

49–52 Find the volume of the parallelepiped spanned by the indicated vectors.

49. $\mathbf{u}=\langle 2,-1,3\rangle, \quad \mathbf{v}=\langle 3,0,4\rangle, \quad \mathbf{w}=\langle -1,1,-2\rangle$

50. $\mathbf{u}=\langle 1,2,3\rangle, \quad \mathbf{v}=\langle 2,-1,4\rangle, \quad \mathbf{w}=\langle -1,0,3\rangle$

51. $\mathbf{u}=\langle 1,1,3\rangle, \quad \mathbf{v}=\langle 1,3,1\rangle, \quad \mathbf{w}=\langle 3,1,1\rangle$

52. $\mathbf{u}=\langle 1,0,-3\rangle, \quad \mathbf{v}=\langle 0,-5,2\rangle, \quad \mathbf{w}=\langle 3,1,1\rangle$

53. In light of Exercises 49–52, state a condition in terms of the triple scalar product for three vectors to be coplanar (that is, to lie on the same plane).

54–55 Use the condition you found in Exercise 53 to determine whether the vectors are coplanar.

54. $\mathbf{u}=\langle 2,-1,3\rangle, \quad \mathbf{v}=\langle -1,2,3\rangle, \quad \mathbf{w}=\langle 3,2,-1\rangle$

55. $\mathbf{u}=\langle 1,5,1\rangle, \quad \mathbf{v}=\langle -2,-4,0\rangle, \quad \mathbf{w}=\langle -3,-15,-3\rangle$

56–57 Use Exercises 53–55 to determine whether the given points are coplanar.

56. $A(0,0,0), \quad B(1,3,4), \quad C(-1,-2,-3), \quad D(1,1,1)$

57. $A(1,5,0),\quad B(2,4,-1),\quad C(0,3,0),\quad D(4,2,-3)$

58. Use the cross product to prove the following well-known formula from trigonometry.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

(**Hint:** See the hint given in Exercise 102 of Section 11.3. Turn the unit vectors into three-dimensional vectors as in Example 4, and interpret their cross product.)

59. A bolt is tightened by an 18-pound force at the end of a 10-inch wrench at an angle of $\theta = 60°$. What is the magnitude of the torque applied to the bolt at the pivot point?

60. Repeat Exercise 59 if the length of the wrench is 16 inches and a rotating force of 15 pounds is applied at $\theta = 45°$.

61. The force **F** exerted by the uniform magnetic field with induction vector **B** on a wire carrying current **I** obeys the vector equation

$$\mathbf{F}=l\,\mathbf{I}\times\mathbf{B},$$

where l is the length of the wire. (The standard SI unit for **B** is the tesla (T). If, in addition, we measure **I** in amperes (A), the above equation will return **F** in newtons (N).) Find the magnitude of the force experienced by an 8 cm wire in a uniform magnetic field of **B** = 2 T if it carries a current of 0.3 A and the angle between the wire and **B** is $\theta = 30°$.

62. The force **F** experienced by a charged particle q moving at velocity $\mathbf{v}\ \text{m/s}$ in the uniform magnetic field **B** obeys the vector equation

$$\mathbf{F}=q\,\mathbf{v}\times\mathbf{B},$$

where (just like in Exercise 61) we obtain the force in newtons (N) if we measure **B** in teslas (T), q in coulombs (C), and **v** in meters per second (m/s). Find the magnitude of the force experienced by an electron moving in a uniform magnetic field of **B** = 0.001 T at $\mathbf{v}=1.2\cdot10^{6}\ \text{m/s}$ if the velocity vector and **B** form a 25° angle. Note that the magnitude of the charge carried by an electron, or the *elementary charge*, is $e\approx1.6\cdot10^{-19}$ C.

63. Use an appropriate property of the cross product (see Exercises 7–13) to verify the following vector equation.

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v})=\mathbf{0}$$

64. Prove that if vectors **s**, **u**, **v**, and **w** are coplanar (i.e., contained by the same plane), then

$$(\mathbf{s}\times\mathbf{u})\times(\mathbf{v}\times\mathbf{w})=\mathbf{0}.$$

65. Let A, B, C, and D be four distinct points in three-dimensional Cartesian space. Prove that $\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\times\overrightarrow{AD}$ is parallel to the plane containing A, B, and C.

66. Find the equation of the plane through the points $(-5,0,4)$, $(2,2,2)$, and $(6,-1,3)$. (**Hint:** See Exercise 60 of Section 11.3 and Exercise 24 of this section.)

67. Repeat Exercise 66 for the plane through $\left(\frac{1}{2},-1,-1\right)$, $\left(\frac{5}{2},3,\frac{7}{2}\right)$, and $\left(2,\frac{1}{2},0\right)$.

68.* (Section 11.3 Exercise 86 revisited) Prove that the distance d from a point Q to a line containing the points R and S is

$$d=\frac{\left|\overrightarrow{RS}\times\overrightarrow{RQ}\right|}{\left|\overrightarrow{RS}\right|}.$$

69.* (Section 11.3 Exercise 88 revisited) Prove that the distance d from a point Q to a plane containing the points R, S, and T is

$$d=\frac{\left|\left(\overrightarrow{RS}\times\overrightarrow{RT}\right)\cdot\overrightarrow{RQ}\right|}{\left|\overrightarrow{RS}\times\overrightarrow{RT}\right|}.$$

70–73. Use Exercises 68–69 to find a second solution for each of Exercises 90–93 of Section 11.3.

74. For vectors **u**, **v**, and **w** in three-dimensional Cartesian space, prove *Lagrange's identity*.

$$|\mathbf{u}\times\mathbf{v}|^{2}=|\mathbf{u}|^{2}|\mathbf{v}|^{2}-(\mathbf{u}\cdot\mathbf{v})^{2}$$

75.* A rectangular tetrahedron is one with a vertex such that any pair of incident edges form right angles. (Such is obtained by "chopping a corner off" a cube.) Denoting the areas of the faces containing a right angle by a, b, and c, respectively, and that of the fourth face by d, prove the following "three-dimensional generalization" of the Pythagorean Theorem.

$$a^{2}+b^{2}+c^{2}=d^{2}$$

(**Hint:** Place the tetrahedron appropriately into the three-dimensional Cartesian system, and use the techniques of this section to find the areas of its faces.)

76. Suppose that the nonzero vectors **u**, **v**, and **w** satisfy

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=\mathbf{0}.$$

Prove that **v** and **w** are parallel, or both are perpendicular to **u**.

77. Prove: $(\mathbf{s}\times\mathbf{u})\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}\mathbf{s}\cdot\mathbf{v} & \mathbf{u}\cdot\mathbf{v}\\ \mathbf{s}\cdot\mathbf{w} & \mathbf{u}\cdot\mathbf{w}\end{vmatrix}.$

78–81 Determine whether the given expression is a vector, a scalar, or nonsense.

78. $\mathbf{u}\times(|\mathbf{u}|\mathbf{v})$

79. $(\mathbf{u}\cdot\mathbf{v})\times|\mathbf{v}|$

80. $(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|$

81. $|\mathbf{v}|(\mathbf{u}\times\mathbf{v})$

Concept Check

82–94 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

82. $\mathbf{u}\times\mathbf{v}=\mathbf{v}\times\mathbf{u}$

83. $\mathbf{u}\times\mathbf{v}=(-\mathbf{u})\times(-\mathbf{v})$

84. $(-\mathbf{u})\times\mathbf{u}=\mathbf{0}$

85. $|\mathbf{u}|\times\mathbf{v}=\mathbf{v}\times|\mathbf{u}|$

86. If $(\mathbf{u}\times\mathbf{v})+\mathbf{w}=\mathbf{0}$, then $(\mathbf{u}\times\mathbf{v})=-\mathbf{w}$.

87. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}$

88. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$

89. $\mathbf{u}\times\mathbf{u}=|\mathbf{u}|^2$

90. If $|\mathbf{u}|<|\mathbf{v}|$ and $\mathbf{w}\neq\mathbf{0}$, then $\mathbf{u}\times\mathbf{w}<\mathbf{v}\times\mathbf{w}$.

91. $(\mathbf{u}-\mathbf{v})\times(\mathbf{u}+\mathbf{v})=2(\mathbf{u}\times\mathbf{v})$

92. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=0$

93. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{v})=0$

94. If $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$ and $\mathbf{u}\neq\mathbf{0}$, then $\mathbf{v}=\mathbf{w}$.

11.4 Technology Exercises

95–96. Use a computer algebra system or programmable calculator to write a program that decides whether four points (given by their coordinates) in three-dimensional space lie in the same plane. Use your program to check your answers for Exercises 56–57.