11.3 Exercises

1–10 Find the dot product of **u** and **v**.

1. $\mathbf{u} = \left\langle 2, \frac{1}{3} \right\rangle, \quad \mathbf{v} = \left\langle \frac{5}{2}, -3 \right\rangle$

2. $\mathbf{u} = \langle 3, -1 \rangle, \quad \mathbf{v} = \langle 0, 0 \rangle$

3. $\mathbf{u} = \langle 2, 0, 1 \rangle, \quad \mathbf{v} = \langle 1, 1, -3 \rangle$

4. $\mathbf{u} = \left\langle \frac{4}{3}, -1, \frac{2}{5} \right\rangle, \quad \mathbf{v} = \left\langle -1, 6, -\frac{5}{3} \right\rangle$

5. $\mathbf{u} = \langle 4s, -2, 2s \rangle, \quad \mathbf{v} = \langle 3, 5, -6 \rangle$

6. $\mathbf{u} = \langle -5s, s, 2 \rangle, \quad \mathbf{v} = \langle 2t, 4t, -1 \rangle$

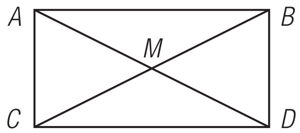
7. $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

8. $\mathbf{u} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{5}{3}\mathbf{k}, \quad \mathbf{v} = 8\mathbf{i} - 5\mathbf{j} - 9\mathbf{k}$

9. $|\mathbf{u}| = 4, \quad |\mathbf{v}| = 3\sqrt{2}, \quad$ their angle is $45°$

10. $|\mathbf{u}| = 2.5, \quad |\mathbf{v}| = 5, \quad$ their angle is $2\pi/3$

11–13 Suppose that the length and width of the rectangle below are $2\sqrt{3}$ and 2 units, respectively. Find the indicated dot product.



11. $\overrightarrow{CM} \cdot \overrightarrow{MD}$

12. $\overrightarrow{CA} \cdot \overrightarrow{MD}$

13. $\overrightarrow{CA} \cdot \left(\overrightarrow{CM} + \overrightarrow{MD} \right)$

14–21 Find the angle between the given vectors.

14. $\mathbf{u} = \left\langle 2, -\sqrt{5} \right\rangle, \quad \mathbf{v} = \langle -4, 3 \rangle$

15. $\mathbf{u} = \langle -1, 1 \rangle, \quad \mathbf{v} = \langle 3, 3 \rangle$

16. $\mathbf{u} = \mathbf{i} - 2\mathbf{j}, \quad \mathbf{v} = 3\mathbf{i} + \mathbf{j}$

17. $\mathbf{u} = (\cos 32°)\mathbf{i} + (\sin 32°)\mathbf{j},$
$\mathbf{v} = -(\cos 87°)\mathbf{i} + (\sin 87°)\mathbf{j}$

18. $\mathbf{u} = \langle 4, -2, 1 \rangle, \quad \mathbf{v} = \langle 2, -1, 5 \rangle$

19. $\mathbf{u} = \langle -1, 1, 2 \rangle, \quad \mathbf{v} = \langle 3, 2, 0 \rangle$

20. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

21. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = \mathbf{i} - \frac{3}{2}\mathbf{j} - \sqrt{3}\,\mathbf{k}$

22–26 Prove the indicated property of the dot product. (**Hint:** Start by representing vectors **u**, **v**, and **w** in component form.)

22. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

23. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

24. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

25. $\mathbf{0} \cdot \mathbf{u} = 0$

26. $a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v}),\ a$ is a scalar

27. Prove that for any positive c,
$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot (c\mathbf{v})}{|c\mathbf{v}|}.$$
(**Hint:** Use the properties of the dot product.)

28. Prove the equation
$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$
for the two cases $\theta = 0$ and $\theta = \pi$. (**Hint:** Use the fact that $\mathbf{v} = c\mathbf{u}$ for some constant c in these two cases, with c positive if $\theta = 0$ and c negative if $\theta = \pi$.)

29–40 Determine whether the given vectors are parallel, orthogonal, or neither.

29. $\langle 2, 6 \rangle$ and $\langle -1, -3 \rangle$

30. $\langle 6, 4 \rangle$ and $\langle -2, 3 \rangle$

31. $\langle 2, -3 \rangle$ and $\left\langle -\frac{1}{2}, \frac{1}{3} \right\rangle$

32. $\langle s, 2t \rangle$ and $\left\langle -3t, \frac{3}{2}s \right\rangle$

33. $\langle 1, 0, -7 \rangle$ and $\left\langle 0, -\frac{5}{3}, 0 \right\rangle$

34. $\langle -s, 5s, 2s \rangle$ and $\left\langle \frac{3}{2}s, \frac{-15}{2}s, -3s \right\rangle$

35. $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} - \mathbf{i}$

36. $\mathbf{i} + \mathbf{j}$ and $-\frac{3}{7}\mathbf{k}$

37. $\frac{1}{2}\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + \frac{3}{2}\mathbf{j} - \mathbf{k}$

38. $-\mathbf{i} + \frac{2}{3}\mathbf{j} - 4\mathbf{k}$ and $5\mathbf{i} + 2\mathbf{j} - \frac{5}{4}\mathbf{k}$

39. $\langle \cos\theta, \sin\theta \rangle$ and $\langle -\sin\theta, \cos\theta \rangle$

40. $\langle \cos(90° - \theta), -\cos\theta, 1 \rangle$ and $\langle -\sin\theta, \cos\theta, 1 \rangle$

41–44 Use the vector method to decide which of the following are true of ΔABC: **a.** acute, **b.** obtuse, **c.** isosceles, **d.** equilateral, **e.** right triangle. (**Hint:** Determine interior angles.)

41. $A(1,2,5),\ \ B(-1,2,-4),\ \ C(8,8,5)$

42. $A(-4,5,6),\ \ B(-2,4,13),\ \ C(-1,-3,4)$

43. $A(2,-1,1),\ \ B(6,2,0),\ \ C(3,1,5)$

44. $A(3,-4,1),\ \ B(1,-3,4),\ \ C(0,-6,2)$

45. Find the angles determined by the two diagonals of the quadrilateral with vertices $\left(1,2,\frac{1}{3}\right)$, $(0,-1,0)$, $\left(3,1,-\frac{4}{3}\right)$, and $(2,-3,-2)$.

46–49 Determine the value of the parameter so that the vectors are orthogonal.

46. $\left\langle s,\frac{1}{2}s,2\right\rangle$ and $\langle -5,2s,3\rangle$

47. $\langle 2,s,-3s\rangle$ and $\langle s,1,1\rangle$

48. $\langle t,-2,t\rangle$ and $\langle t,t,t^2\rangle$

49. $\langle 4,s,-2\rangle$ and $\langle 4,5t,8\rangle$

50. Find a vector of length $\sqrt{3}$ in $\mathbb{R}^3$ that is orthogonal to both $\langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$.

51. Use vectors to show that a parallelogram is a rhombus if and only if its diagonals are perpendicular.

52. Prove that if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

53. Thales' Theorem states that any point on a circle determines a right triangle with the endpoints of any diameter not containing the point. Use vectors to prove Thales' Theorem. (**Hint:** It is enough to prove the statement for the unit circle centered at the origin.)

54. Find an equation of the line that contains the point (a,b) and is perpendicular to the vector $\mathbf{n}=\langle n_1,n_2\rangle$ ($\mathbf{n}$ is called a normal vector to the line). (**Hint:** Notice that for any point (x,y) on the line, the vector $\langle x-a,y-b\rangle$ must be perpendicular to $\mathbf{n}$, so their dot product is 0.)

55. Use Exercise 54 to derive the point-slope form of the equation of a line of slope m through the point (a,b). (**Hint:** Notice that $\mathbf{n}=\langle -m,1\rangle$ is a normal vector.)

56. Use normal vectors to determine the angle between the lines with slopes $m = 1$ and $m = 3$, respectively. (**Hint:** Note that the angle between the lines is the same as that between their respective normal vectors. See the hint given in Exercise 55.)

57–58 Find a unit vector that is normal (or perpendicular) to the given line.

57. $x-3y=7$

58. $y=\frac{1}{2}x-5$

59. Generalize Exercise 54 to find an equation of the plane in $\mathbb{R}^3$ that contains the point (a,b,c) and is perpendicular to $\mathbf{n}=\langle n_1,n_2,n_3\rangle$. (The vector $\mathbf{n}$ is called a normal vector to the plane. We will elaborate on this approach in Section 11.5.)

60. Use Exercise 59 to find the equation of the plane through $(2,-4,1)$ that is perpendicular to $\mathbf{n}=\langle 5,-3,1\rangle$.

61–62 Find a normal vector to the given plane.

61. $x-4y-2z=7$

62. $2x+y-5z=1$

63. Use the results of Exercises 61 and 62 to find the angle between the planes $x-4y-2z=7$ and $2x+y-5z=1$. (**Hint:** The angle between two planes is the same as that between their respective normal vectors.)

64–67 Find the direction angles of the given vector.

64. $\langle 1,0,-1\rangle$

65. $\langle 2,-1,4\rangle$

66. $\left\langle -\frac{1}{3},2,-\frac{2\sqrt{3}}{3}\right\rangle$

67. $\left\langle \frac{1}{2},-4,-\frac{5}{3}\right\rangle$

68. Suppose the first and second direction angles of a vector in the first octant are $\pi/6$ and $\pi/3$. Find the third direction angle.

69. Prove the *Cauchy-Schwarz Inequality*: For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$$|\mathbf{u}\cdot\mathbf{v}|\le|\mathbf{u}||\mathbf{v}|.$$

Under what conditions does equality hold? (Note that on the left-hand side, $|\cdot|$ means the absolute value of a scalar, while the right-hand side is the product of the magnitudes of the vectors.)

70. Use the Cauchy-Schwarz Inequality (Exercise 69) to prove the famous *Triangle Inequality*: For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$$|\mathbf{u}+\mathbf{v}|\le|\mathbf{u}|+|\mathbf{v}|.$$

Under what conditions does equality hold? (Note that this inequality states that the sum of the lengths of two vectors never exceeds the sum of their individual lengths. **Hint:** Estimate $|\mathbf{u}+\mathbf{v}|^2$ by writing $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$. Then use the properties of the dot product and the Cauchy-Schwarz Inequality.)

71. What can you say about $\mathbf{v}$ if $|\mathbf{u}|=2$, $|\mathbf{v}|=4$, and $|\mathbf{u}\cdot\mathbf{v}|$ is maximum?

72. Use Exercise 70 to prove the following.

$$\big||\mathbf{u}|-|\mathbf{v}|\big| \le |\mathbf{u}-\mathbf{v}|$$

(This is often called the "left-hand part of the Triangle Inequality," or the "Reverse Triangle Inequality.")

73. Prove the following so-called *Parallelogram Law*. Can you give a reason for its name?

$$|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2.$$

74–83 Decompose **u** into a sum of two vectors, one parallel to **v** and one perpendicular to **v**.

74. $\mathbf{u}=\langle 3,-1\rangle,\quad \mathbf{v}=\langle 1,7\rangle$

75. $\mathbf{u}=\left\langle 2,\frac{1}{3}\right\rangle,\quad \mathbf{v}=\left\langle \frac{5}{2},-3\right\rangle$

76. $\mathbf{u}=\langle 2,0,1\rangle,\quad \mathbf{v}=\langle 1,1,-3\rangle$

77. $\mathbf{u}=\left\langle \frac{4}{3},-1,\frac{2}{5}\right\rangle,\quad \mathbf{v}=\left\langle -1,6,-\frac{5}{3}\right\rangle$

78. $\mathbf{u}=\langle 4s,-2,2s\rangle,\quad \mathbf{v}=\langle 3,5,-6\rangle$

79.* $\mathbf{u}=\langle -5s,s,2\rangle,\quad \mathbf{v}=\langle 2t,4t,-1\rangle$

80. $\mathbf{u}=\mathbf{i}+3\mathbf{j}-2\mathbf{k},\quad \mathbf{v}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$

81. $\mathbf{u}=\frac{1}{2}\mathbf{i}+\mathbf{j}+\frac{5}{3}\mathbf{k},\quad \mathbf{v}=8\mathbf{i}-5\mathbf{j}-9\mathbf{k}$

82. $|\mathbf{u}|=4,\quad |\mathbf{v}|=3\sqrt{2},\quad$ their angle is $45°$

83. $|\mathbf{u}|=2.5,\quad |\mathbf{v}|=5,\quad$ their angle is $2\pi/3$

84. For $\mathbf{v}=\langle 2,-1\rangle$, find a vector **u** such that

a. $|\text{proj}_{\mathbf{v}}\mathbf{u}|=\frac{5}{2}$, **b.** $|\text{proj}_{\mathbf{u}}\mathbf{v}|=\frac{5}{2}$.

(Answers will vary.)

85. For $\mathbf{v}=\langle -3,0,1\rangle$, find a vector **u** such that $|\text{proj}_{\mathbf{v}}\mathbf{u}|=\frac{5}{2}$.

86.* Prove that the distance d from a point $Q(x_0,y_0)$ to a line $ax+by=c$ is

$$d=\frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}.$$

(**Hint:** Revisit Exercises 59 and 60 to identify a normal vector of the line, then pick a point $P(x,y)$ on the line, and consider $\text{proj}_{\mathbf{n}}\overrightarrow{PQ}$.)

87.* Prove that the distance d between parallel lines $ax+by=c_1$ and $ax+by=c_2$ is

$$d=\frac{|c_2-c_1|}{\sqrt{a^2+b^2}}.$$

88.* Generalize your solution to Exercise 86 to three dimensions to show that the distance d from a point $Q(x_0,y_0,z_0)$ to a plane $ax+by+cz=f$ is

$$d=\frac{|ax_0+by_0+cz_0-f|}{\sqrt{a^2+b^2+c^2}}.$$

89.* Generalize Exercise 87 to show that the distance d between parallel planes $ax+by+cz=f_1$ and $ax+by+cz=f_2$ is

$$d=\frac{|f_2-f_1|}{\sqrt{a^2+b^2+c^2}}.$$

90–93 Use the formulas from Exercises 86–89 to find the indicated distance.

90. The distance between the point $(1,2)$ and the line $x-y=4$

91. The distance between the lines $2x+3y=2$ and $2x+3y=-5$

92. The distance between the point $(1,2,3)$ and the plane $x-3y+z=2$

93. The distance between the planes $x-2y+5z=1$ and $x-2y+5z=7$

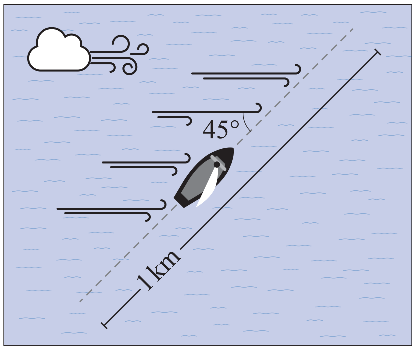
94–95 Find the work done by the force **F** as it moves an object from P to Q. (Suppose **F** is measured in pounds and a unit distance is 1 foot.)

94. $\mathbf{F}=-2\mathbf{i}-\mathbf{j}+5\mathbf{k}$ from $P(1,2,3)$ to $Q(4,-7,6)$

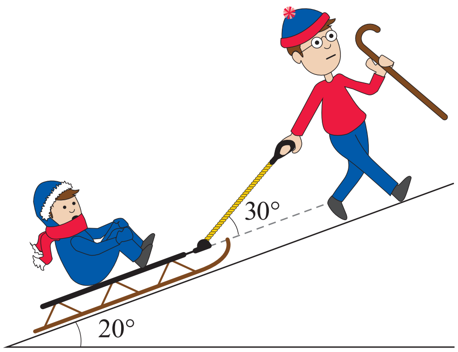
95. $\mathbf{F}=\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ from $P(-7,1,-4)$ to $Q(-3,-6,2)$

96. In order to close a curtain, a hotel guest pulls a rod at an angle of 45° with a constant force of 10 pounds. Find the work done if the curtain moves 5 feet to its closed position.

97. A sailboat is propelled a distance of 1 km by a wind that makes a 45° angle with the boat's direction of travel. Find the work done by the wind if its force is 2000 newtons.



98. A father is pulling his little son in a sled up a 20° slope that is 20 meters long. The rope makes a 30° angle with the surface of the slope and the combined weight of the child and the sled is 200 newtons. (Ignore friction and any acceleration of motion.)



a. Find the work done by the father on the sled.

b. Find the force of tension in the rope.

99.* Repeat Exercise 98 under the assumption that the coefficient of friction is $\mu = 0.13$.

100. Use vectors to find the angle between the diagonal of a cube and

a. one of its edges,

b. the diagonal of one of the faces.

101.* Prove that the points $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$ determine a regular tetrahedron and find the angle any edge makes with the face that doesn't contain it.

102. Use two-dimensional unit vectors to prove the following well-known formula from trigonometry:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(**Hint:** Considering two unit vectors, represent them as in Exercise 17, and interpret their dot product.)

103. A company manufactures three different products, producing n_j of each during a given production cycle ($j = 1, 2, 3$). Accordingly, the total production of a given cycle can be arranged into a three-dimensional production vector: $\mathbf{p} = \langle p_1, p_2, p_3 \rangle$. If the selling price of the j^{th} product is s_j dollars each, the price vector can similarly be defined as $\mathbf{s} = \langle s_1, s_2, s_3 \rangle$. Interpret the dot product $\mathbf{p} \cdot \mathbf{s}$ for a given production cycle.

Concept Check

104–120 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

104. $\langle 1,3,0 \rangle \cdot \langle -2,1 \rangle = 1$

105. $|\mathbf{u}|\mathbf{v} = \mathbf{v}|\mathbf{u}|$

106. $|\mathbf{u}|\mathbf{v} = \mathbf{u}|\mathbf{v}|$

107. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

108. $|\mathbf{u}| \cdot (\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (|\mathbf{u}|\mathbf{w})$

109. $|\mathbf{u}|(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (|\mathbf{u}|\mathbf{w})$

110. If $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w} = 0$, then $(\mathbf{u} \cdot \mathbf{v}) = -\mathbf{w}$.

111. $(\mathbf{u} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{x}) = [(\mathbf{w} \cdot \mathbf{x})\mathbf{u}] \cdot \mathbf{v} = \mathbf{u} \cdot [(\mathbf{w} \cdot \mathbf{x})\mathbf{v}]$

112. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.

113. $|\mathbf{u}|(\mathbf{v} + \mathbf{w}) = |\mathbf{u}|\mathbf{v} + |\mathbf{u}|\mathbf{w}$

114. $\mathbf{u} \cdot \mathbf{v} < 0$ if and only if the angle between **u** and **v** is obtuse.

115. $(\text{proj}_{\mathbf{v}}\mathbf{u} - \mathbf{u}) \cdot \text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{0}$

116. If **u** is orthogonal to both **v** and **w**, then **u** is orthogonal to **v** + **w**.

117. If $|\text{proj}_{\mathbf{v}}\mathbf{u}| = |\mathbf{u}|$, then **u** and **v** are parallel.

118. If $|\text{proj}_{\mathbf{v}}\mathbf{u}| = 0$, then **u** and **v** are orthogonal.

119. If $|\mathbf{u}| < |\mathbf{v}|$, then $\mathbf{u} \cdot \mathbf{w} < \mathbf{v} \cdot \mathbf{w}$.

120. If $|\text{proj}_{\mathbf{v}}\mathbf{u}| = |\text{proj}_{\mathbf{u}}\mathbf{v}|$, then $|\mathbf{u}| = |\mathbf{v}|$.