$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	$\lvert x \rvert < \infty$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	$\lvert x \rvert < \infty$
$\ln x$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$0 < x \le 2$

Table 1

10.8 Exercises

1. Find the Maclaurin polynomial of order 4 generated by $f(x)$ given that $f(0)=4$, $f'(0)=2$, $f''(0)=-1$, $f'''(0)=3$, and $f^{(4)}(0)=1$.

2. Find the Taylor polynomial of order 4 generated by $f(x)$ about $a=2$ given that $f(2)=1$, $f'(2)=-3$, $f''(2)=0$, $f'''(2)=5$, and $f^{(4)}(2)=-2$.

3–10 Determine the Taylor polynomials generated by the given function about the indicated point. Find the radius of convergence.

3. $f(x)=2x^4-x^3+5x^2+3x-7;\quad a=0$
4. $f(x)=e^{-x^2/2};\quad a=0$
5. $f(x)=e^x;\quad a=1$
6. $f(x)=\sin x;\quad a=\pi/2$
7. $f(x)=\frac{1}{x+4};\quad a=1$
8. $f(x)=\ln x;\quad a=1$
9. $f(x)=\frac{1}{x^2};\quad a=2$
10. $f(x)=\tanh^{-1} x;\quad a=0$

11–29 Use the Taylor series (or Maclaurin series if the center is not specified) we discussed so far in the text (see Table 1) to determine that of the given function. In each case, find the radius of convergence.

11. $f(x)=\cos(x^2)$
12. $f(x)=\cos^2 x=\frac{1+\cos 2x}{2}$
13. $f(x)=\sin x\cos x=\frac{\sin 2x}{2}$
14. $f(x)=x^3e^{-2x}$
15. $f(x)=\cosh x$
16. $f(x)=xe^x;\quad a=1$
17. $f(x)=\cos\sqrt{x}$
18. $f(x)=\frac{x+1}{x-1}$
19. $f(x)=(2x^3-x)e^x$
20. $f(x)=\sinh x$
21. $f(x)=\ln(1+x^2)$
22. $f(x)=\frac{x}{1+x^4}$
23. $f(x)=\frac{1}{x^2};\quad a=3$
24. $f(x)=\ln\frac{1+x}{1-x};\quad a=0$
25. $f(x)=\frac{2x+1}{x^2+x-6}$
26. $f(x)=\frac{1}{(x^2+1)^2}$
27. $f(x)=\frac{x^2}{1-x^2}$
28. $f(x)=\tan^{-1} x;\quad a=0$
29. $f(x)=x\cos(x^{3/2});\quad a=0$

30–35 Use the definition to find the first five nonzero terms of the Taylor series generated by the given function about the indicated point.

30. $f(x)=\tan x;\quad a=\pi/4$
31. $f(x)=\arctan x;\quad a=1$
32. $f(x)=\sec x;\quad a=0$

33. $f(x)=e^{\cos x};\quad a=0$

34. $f(x)=e^{\sin x};\quad a=0$

35. $f(x)=2^x;\quad a=1$

36–47 Find the first five nonzero terms of the Maclaurin series generated by the indicated function by using operations on familiar series (try not to use the definition). In Exercises 40–42, try long division.

36. $f(x)=\sin x+\cos x$

37. $f(x)=\dfrac{e^x}{1-x}$

38. $f(x)=e^x\cos x$

39. $f(x)=e^{-2x}\sin x$

40. $f(x)=\dfrac{\cos x}{x+1}$

41. $f(x)=\tanh x$

42. $f(x)=\tan x$

43. $f(x)=\ln(1+x)\sin x$

44. $f(x)=x\sec 2x$

45. $f(x)=\ln(1-x)\cos x$

46. $f(x)=\cos\left(x^2+2x\right)$

47. $f(x)=(x-1)^2\cos\pi x$

48–57 Find the Taylor series approximation of the given function value or definite integral that guarantees the indicated accuracy. How many (nonzero) terms did you use? (**Hint:** See Example 8 and note the nonelementary integrals that we cannot evaluate exactly.)

48. $\ln 1.2;\quad \text{error}\le 10^{-4}$ (**Hint:** Use the Taylor series expansion of $\ln(x+1)$.)

49. $\dfrac{1}{e};\quad \text{error}\le 10^{-4}$

50. $\sin 1;\quad \text{error}\le 10^{-5}$

51. $\int_0^1 e^{-x^4}\,dx;\quad \text{error}\le 0.001$

52. $\int_0^1 \cos\left(x^3\right)dx;\quad \text{error}\le 10^{-4}$

53. $\int_0^1 \sin\left(x^2\right)dx;\quad \text{error}\le 10^{-6}$

54. $\int_0^1 \dfrac{\sin x}{x}\,dx;\quad \text{error}\le 10^{-6}$

55. $\int_0^1 \tan^{-1}\left(x^2\right)dx;\quad \text{error}\le 0.01$

56. $\int_0^1 x^2\sin\left(x^2\right)dx;\quad \text{error}\le 10^{-6}$

57. $\int_0^1 \dfrac{1}{\sqrt{x^4+1}}\,dx;\quad \text{error}\le 0.01$

58–60 Prove that the Maclaurin series converges to the function by showing that the error $r_n(x)$ satisfies $\lim_{n\to\infty} r_n(x)=0$.

58. $e^{-x^2}=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{n!}$

59. $\cos x=\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{(2n)!}$

60. $\ln(x+1)=\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}x^n}{n},\quad |x|<1$

61–64 Use Taylor series to find the indicated limit.

61. $\lim_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x}$

62. $\lim_{x\to 0}\dfrac{2x^3-1+\cos x}{x^5}$

63. $\lim_{x\to 0}\dfrac{\tan^{-1}x-\sin\left(x^2\right)}{x}$

64. $\lim_{x\to 0}\dfrac{\ln(x+1)}{x\sin x}$

65. Use power series to show that $\lim_{x\to 0}\dfrac{\sin x}{x}=1$.

66. Find the Taylor series expansion of the function $f(x)=3x^3-5x^2+2x-7$ about $a=2$. What can you tell from your answer?

67. Use the Maclaurin series expansion of $f(x)=1/(1-x)$ and differentiation to find $\sum_{n=1}^{\infty}\left(n/2^{n-1}\right)$.

68.* Calculate the first few terms of the Maclaurin series expansion of $f(x)=\arcsin x$; then notice the apparent pattern and derive the general form of the series. What is the radius of convergence?

69.* Suppose that the function f has derivatives of all orders throughout an open interval I and there is an $M>0$ such that $\left|f^{(n)}(x)\right|\le M$ for all $n\in\mathbb{N}$ over I. Prove that if $a\in I$, then the Taylor series generated by $f(x)$ about a converges to f at any $x\in I$.

70. Use Exercise 68 to prove that if $f(x)=\sinh x$ or $f(x)=\cosh x$, then the Taylor series expansion of f about any $a\in I$ converges to f at every real number x.

71.* Find the value of the 162nd derivative of $f(x)=\tan^{-1}x$ at $x=0$. (**Hint:** Examine what happens to the Maclaurin series after repeated differentiation and substitution of $x=0$.) What is the 163rd derivative at 0?

72. Use the Maclaurin series expansion of $f(x)=\left(x+x^2\right)e^x$ to prove $\sum_{n=1}^{\infty}\left(n^2/n!\right)=2e$.

73.* Use an appropriate Maclaurin series expansion to prove $\sum_{n=1}^{\infty}\left(n^3/n!\right)=5e$. (**Hint:** See Exercise 72.)