

So as  $n \rightarrow \infty$ ,  $a_{n+1}/a_n$  oscillates between values approaching 0 (for odd  $n$ ) and values growing unboundedly large (for even  $n$ )—in other words,  $\lim_{n \rightarrow \infty} (a_{n+1}/a_n)$  fails to exist in a fairly extreme fashion. But

$$\sqrt[n]{a_n} = \begin{cases} n^{1/n}/2 & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even} \end{cases}$$

and  $n^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ , so

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$$

and the series converges by the Root Test.

## 10.5 Exercises

**1–34** Use the Ratio Test to determine whether the series converges or diverges.

1.  $\sum_{n=0}^{\infty} \frac{1}{n!}$

2.  $\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n$

3.  $\sum_{n=0}^{\infty} \frac{5^n}{n!}$

4.  $\sum_{n=1}^{\infty} \frac{\pi^n}{n^\pi}$

5.  $\sum_{n=0}^{\infty} \frac{3}{(2n)!}$

6.  $\sum_{n=1}^{\infty} \frac{2^n}{n}$

7.  $\sum_{n=1}^{\infty} \frac{n^5}{e^n}$

8.  $\sum_{n=0}^{\infty} \frac{3}{(2n+1)!}$

9.  $\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$

10.  $\sum_{n=2}^{\infty} \frac{2n}{(n-1)^2}$

11.  $\sum_{n=0}^{\infty} \frac{3n+2}{5^{3n+2}}$

12.  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

13.  $\sum_{n=1}^{\infty} \frac{(3n)!}{4^{n-1}5^n}$

14.  $\sum_{n=0}^{\infty} \frac{n!}{6^n}$

15.  $\sum_{n=1}^{\infty} \frac{n}{5^n}$

16.  $\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{(n+1)!}$

17.  $\sum_{n=1}^{\infty} \frac{4^n}{(n+4)n}$

18.  $\sum_{n=1}^{\infty} \frac{n}{\pi^{2n+1}}$

19.  $\sum_{n=1}^{\infty} \frac{n^2-4}{2^n}$

20.  $\sum_{n=0}^{\infty} \frac{(n!)4^n}{(4n+1)!}$

21.  $\sum_{n=2}^{\infty} \frac{2^n}{n(n^2-1)}$

22.  $\sum_{n=1}^{\infty} \frac{n^2}{(\ln 2)^n}$

23.  $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 3)^n}$

24.  $\sum_{n=0}^{\infty} \frac{(3n-2)(3n+2)}{3^n}$

25.  $\sum_{n=0}^{\infty} \frac{(n+3)3^n}{n!}$

26.  $\sum_{n=1}^{\infty} \frac{n^2}{(\ln \pi)^n}$

27.  $\sum_{n=1}^{\infty} \frac{2^{3n-2}}{2n^2+3n}$

28.  $\sum_{n=1}^{\infty} \frac{(3n)!}{n^3}$

29.  $\sum_{n=0}^{\infty} \frac{2^n}{(n+2)^n}$

30.  $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

31.  $\sum_{n=0}^{\infty} \frac{3^n}{2^n+1}$

32.  $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

33.  $\frac{2}{1 \cdot 2 \cdot 3} + \frac{4}{2 \cdot 3 \cdot 4} + \cdots + \frac{2^n}{n(n+1)(n+2)} + \cdots$

34.  $\frac{1}{3} + \frac{1 \cdot 5}{2 \cdot 3 \cdot 9} + \cdots + \frac{1 \cdot 5 \cdots (4n-3)}{3^n (n!)(2n-1)} + \cdots$

**35–38** Verify that the Ratio Test is inconclusive for the series. Then determine the convergence or divergence of the series by some other means.

35.  $\sum_{n=0}^{\infty} \frac{n^2+1}{(n+1)^2}$

36.  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$

37.  $\sum_{n=1}^{\infty} \frac{\arctan n}{n}$

38.  $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n}(n+2)^2}$

**39–62** Use the Root Test to determine whether the series converges or diverges.

39.  $\sum_{n=1}^{\infty} \left(\frac{5}{n}\right)^n$

40.  $\sum_{n=0}^{\infty} \frac{1}{(n+1)^n}$

41.  $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

42.  $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n^n}$

43.  $\sum_{n=1}^{\infty} \left(\frac{3n}{n^2+3}\right)^n$

44.  $\sum_{n=2}^{\infty} \frac{n^{2n}}{(\ln n)^n}$

45.  $\sum_{n=1}^{\infty} \frac{n}{2^{2n}}$

46.  $\sum_{n=2}^{\infty} \frac{e^{2n}}{(\ln n)^n}$

$$47. \sum_{n=0}^{\infty} \frac{2^n}{(n+2)^n}$$

$$48. \sum_{n=1}^{\infty} \left( \frac{n}{2n+3} \right)^n$$

$$49. \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^{n^2}$$

$$50. \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^{-n^2}$$

$$51. \sum_{n=2}^{\infty} \frac{1}{(\ln(\ln n))^n}$$

$$52. \sum_{n=2}^{\infty} \left( \ln \frac{1}{n} \right)^{2n}$$

$$53. \sum_{n=1}^{\infty} n \left( \frac{4}{5} \right)^n$$

$$54. \sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$$

$$55. \sum_{n=2}^{\infty} \left( \frac{\ln n}{n} \right)^n$$

$$56. \sum_{n=1}^{\infty} \left( \frac{2n-1}{3n+2} \right)^n$$

$$57. \sum_{n=2}^{\infty} \frac{n^{n/2}}{(\ln n)^n}$$

$$58. \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$$

$$59. \sum_{n=1}^{\infty} \frac{(n+2)^n}{n^{2n}}$$

$$60. \sum_{n=1}^{\infty} \frac{2}{n^{n+2}}$$

$$61. \sum_{n=1}^{\infty} \sin^n \frac{1}{n^2}$$

$$62. \sum_{n=1}^{\infty} \frac{1}{\left( 2 + \frac{1}{n} \right)^n}$$

**63–66** Verify that the Root Test is inconclusive for the series. Then determine the convergence or divergence of the series by some other means.

$$63. \sum_{n=1}^{\infty} \left( \frac{n}{n+5} \right)^n$$

$$64. \sum_{n=1}^{\infty} \left( \frac{2\sqrt{n}}{4n+1} \right)^{2n}$$

$$65. \sum_{n=1}^{\infty} \left( 1 - \frac{1}{n} \right)^n$$

$$66. \sum_{n=1}^{\infty} \frac{2n}{(n+1)^3}$$

**67–70** Suppose that the series  $\sum a_n$  satisfies the condition  $\lim_{n \rightarrow \infty} (a_{n+1}/a_n) = \frac{1}{2}$ . Decide whether the given series is convergent.

$$67. \sum_{n=1}^{\infty} 3^n a_n$$

$$68. \sum_{n=1}^{\infty} n^2 a_n$$

$$69. \sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n a_n$$

$$70. \sum_{n=1}^{\infty} a_n^2$$

71. Prove that for all exponents  $p$ , the series  $\sum (n^p/2^n)$  is convergent.
72. Prove that  $\sum (p^n/n!)$  converges for all  $p > 0$ .
73. For what positive  $p$ -values does the series  $\sum (p^n/n)$  converge?

## Concept Check

**74–79** Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample. ( $\sum a_n$  is a positive series in each problem.)

74. If  $\sum a_n$  is convergent, then  $\lim_{n \rightarrow \infty} (a_{n+1}/a_n) < 1$ .

75. If  $\sum a_n$  is divergent, then  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} \geq 1$  or the limit doesn't exist.

76. If there is an  $N > 0$  such that  $\sqrt[n]{a_n} < 1$  for all  $n \geq N$ , then  $\sum a_n$  is convergent.

77. The series  $\sum_{n=1}^{\infty} (3^{n^2}/n!)$  is convergent.

78.\* If  $\sum a_n$  satisfies condition a. of the Ratio Test, then it satisfies condition a. of the Root Test.

79.\* If  $\sum a_n$  satisfies the condition a. of the Root Test, then it satisfies condition a. of the Ratio Test.

## 10.5 Technology Exercises

**80.** The following series, discovered by the great Indian mathematician Srinivasa Ramanujan, converges to  $1/\pi$  with amazing speed.

$$\frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(26390n+1103)}{(n!)^4 396^{4n}} = \frac{1}{\pi}$$

- a. Prove that the series is convergent.
- b. Use Ramanujan's series with a graphing utility to approximate  $\pi$ , correct to 31 digits after the decimal. How many terms did you have to use to achieve this accuracy?