Solution

Since

$$\int_n^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{n}\right) = \frac{1}{n},$$

we can make the following estimates.

a. $$s_{10} + \frac{1}{11} \le s \le s_{10} + \frac{1}{10}$$
$$1.55 + 0.09 \le s \le 1.55 + 0.10$$
$$1.64 \le s \le 1.65$$

b. $$s_{1000} + \frac{1}{1001} \le s \le s_{1000} + \frac{1}{1000}$$
$$1.643935 + 0.000999 \le s \le 1.643935 + 0.001$$
$$1.644934 \le s \le 1.644935$$

Since s lies within an interval of width 1×10^{-6}, we could approximate s by the average of the two values, 1.6449345, and know that the approximation is accurate to within $\frac{1}{2}\left(1\times10^{-6}\right) = 5\times10^{-7}$. (The exact value of s is $\pi^2/6$, as shown in Exercise 82 of Section 10.2)

Although the harmonic series diverges, we can still use an integral comparison to gain an appreciation for how slowly the sum grows. As we saw in proving the Integral Test, the relationship $\int_1^n f(x)\,dx \le s_{n-1}$ is valid for any series of positive but decreasing terms.

Example 4 Finding a Lower Bound for a Partial Sum

Find a lower bound for the sum of the first 1 million terms of the harmonic series.

Solution

Using $\int_1^n f(x)\,dx \le s_{n-1}$ with $n = 1{,}000{,}001$ gives us

$$\int_1^{1{,}000{,}001} (1/x)\,dx = \ln(1{,}000{,}001) \approx 13.816.$$

In fact, after a significant amount of computation, a computer algebra system can tell us that $s_{1{,}000{,}000} \approx 14.393$, so the lower bound given by the integral is a very quick and reasonable indication of the rate at which a divergent series of positive decreasing terms grows.

10.3 Exercises

1–32 Use the Integral Test to determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{2}{n+1}$

2. $\sum_{n=1}^{\infty} \frac{1}{3n-2}$

3. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

4. $\sum_{n=1}^{\infty} \frac{2}{n^{5/4}}$

5. $\sum_{n=1}^{\infty} ne^{-n}$

6. $\sum_{n=4}^{\infty} \frac{6}{n^2-9}$

7. $\sum_{n=4}^{\infty} \frac{6}{(n-3)^2}$

8. $\sum_{n=1}^{\infty} \frac{n-1}{n^2-2n+0.75}$

9. $\sum_{n=1}^{\infty} \frac{2\ln n}{n}$

10. $\sum_{n=1}^{\infty} \frac{2\ln n}{n^2}$

11. $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$

12. $\sum_{n=2}^{\infty} \frac{2}{n\ln n}$

13. $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}}$

14. $\sum_{n=1}^{\infty} \frac{n}{(n^2+4)^2}$

15. $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$

16. $\sum_{n=3}^{\infty} \frac{1}{n^2-2n}$

17. $\sum_{n=1}^{\infty} \left(\frac{1}{n}+\frac{1}{2^n}\right)$

18. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n+1}}$

19. $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)^2}$

20. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{3n+1}}$

21. $\sum_{n=1}^{\infty} n^2 e^{-n}$

22. $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}$

23. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

24. $\sum_{n=1}^{\infty} \frac{1}{an+b}, \quad a \neq 0$

25. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

26. $\sum_{n=1}^{\infty} \frac{4n}{n^4+1}$

27. $\sum_{n=1}^{\infty} \frac{4n^3}{n^4+1}$

28. $\sum_{n=1}^{\infty} \left(2n^{-5/4}+\frac{1}{n^3}\right)$

29. $\sum_{n=1}^{\infty} \left(\frac{n}{(n^2+1)^3}+\frac{1}{n^2}\right)$

30. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)^2}$

31. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{4/5}}$

32. $\sum_{n=0}^{\infty} \left(2^{-n}+\left(\frac{e}{3}\right)^n\right)$

33–36 Explain why the Integral Test is not applicable to test the series for convergence.

33. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^3}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

35. $\sum_{n=0}^{\infty} \frac{\cos n}{2^n}$

36. $\sum_{n=0}^{\infty} e^{-\pi n}\cos(\pi n)$

37. Use the Integral Test to show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

diverges when $p \leq 1$ and converges if $p > 1$. (These are called *logarithmic p-series*.)

38–43 Use our estimates immediately preceding Example 3 with the indicated number of terms to find an interval containing s, the sum of the series.

38. $\sum_{n=1}^{\infty} \frac{1}{n^4}$; four terms

39. $\sum_{n=1}^{\infty} \frac{1}{n^3}$; five terms

40. $\sum_{n=1}^{\infty} e^{-n}$; five terms

41. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$; six terms

42. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}(n+1)}$; five terms

43. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$; five terms

44–49 Find the smallest possible value of n to approximate the sum of the given series within the indicated error ε and provide the requested estimate.

44. $\sum_{n=1}^{\infty} \frac{1}{n^4}$; $\varepsilon = 0.005$

45. $\sum_{n=1}^{\infty} \frac{1}{n^3}$; $\varepsilon = 0.005$

46. $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$; $\varepsilon = 0.01$

47. $\sum_{n=1}^{\infty} ne^{-n^2}$; $\varepsilon = 5\times10^{-8}$

48. $\sum_{n=1}^{\infty} \frac{2}{(n+2)[\ln(n+2)]^3}$; $\varepsilon = 0.01$

49. $\sum_{n=1}^{\infty} \frac{3}{1+n^2}$; $\varepsilon = 0.05$

10.3 Technology Exercises

50–54 Use a graphing utility and remainder estimates to approximate the sum of the series with the given error ε. Find the smallest possible value of n you can use and give an approximation with the requested accuracy. (Answers may vary slightly. Recall that formulas for these sums are unavailable; see the discussion preceding Exercise 82 in Section 10.2.)

50. $\sum_{n=1}^{\infty} \frac{1}{n^3}$; $\varepsilon = 10^{-8}$

51. $\sum_{n=1}^{\infty} \frac{1}{n^5}$; $\varepsilon = 5\times10^{-9}$

52. $\sum_{n=1}^{\infty} \frac{1}{n^7}$; $\varepsilon = 10^{-12}$

53. $\sum_{n=1}^{\infty} \frac{1}{n^9}$; $\varepsilon = 10^{-12}$

54. Approximate π to eight decimal places using the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. What value of n did you use? (Answers will vary.)