Squaring both sides, we obtain the equation $L^2 = 2 + L$, or $L^2 - L - 2 = 0$. One of the solutions of this equation is $L = 2$, and the other $(L = -1)$ is extraneous to this problem.

10.1 Exercises

1–4 List the first six terms of the given sequence.

1. $a_n = \dfrac{n}{n^2+1}$ **2.** $a_n = \left(-\dfrac{2}{5}\right)^n$ **3.** $a_n = \dfrac{3}{(n-1)!}$ **4.** $a_n = \dfrac{n(n+1)}{2}\cos(n\pi)$

5–8 Find the first six terms of the given recursively defined sequence.

5. $a_1 = 1, \quad a_n = 2a_{n-1} + 1$ for $n \geq 2$

6. $a_1 = 2, \quad a_2 = 3, \quad a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$

7. $a_1 = 4, \quad a_{n+1} = (n+1)a_n$ for $n \geq 1$

8. $a_1 = 1, \quad a_2 = 1, \quad a_{n+1} = -2a_n + 3a_{n-1}$ for $n \geq 2$

9–12 Recognize the apparent pattern and find an explicit formula for the sequence. (Answers will vary.)

9. $\{2, 6, 12, 20, 30, 42, \ldots\}$

10. $\{2, 6, 18, 54, 162, 486, \ldots\}$

11. $\{9, 6, 1, -6, -15, -26, \ldots\}$

12. $\{2, 4, 7, 11, 16, 22, \ldots\}$

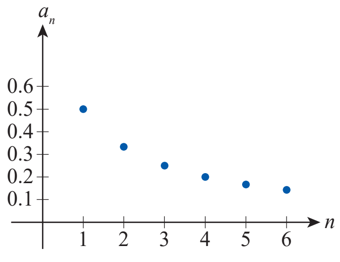
13. Notice that the sequences in Exercises 9–12 are actually not uniquely determined. For example, show that in each case the formula you obtained for a_n and the formula $b_n = a_n + (n-1)\cdots(n-j)$ define two sequences that match for the first j terms and then differ.

14–19 Match the sequence with its graph (labeled A–F).

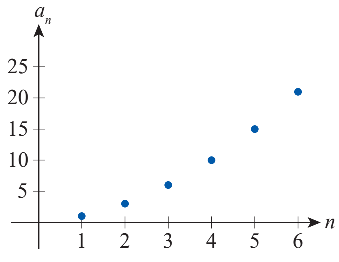
14. $a_n = \dfrac{(-1)^n}{\sqrt{n}}$ **15.** $a_n = \dfrac{n(n+1)}{2}$ **16.** $a_n = (-2)^n$

17. $a_n = \cos(n\pi)$ **18.** $a_n = \dfrac{1}{n+1}$ **19.** $a_n = \dfrac{n}{n+1}$

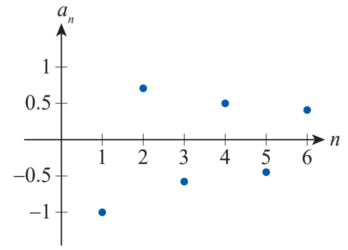
A.



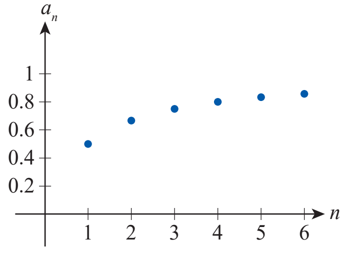
B.



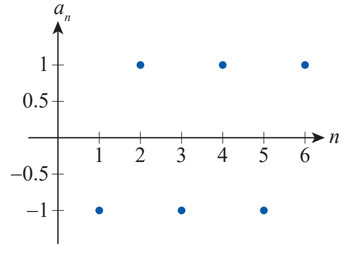
C.



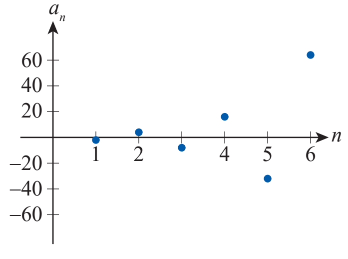
D.



E.



F.



20–23 Use the definition of the limit of a sequence to establish the given statement.

20. $\lim_{n\to\infty} \frac{1}{n^2} = 0$

21. $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

22. $\lim_{n\to\infty} \frac{3n+1}{n+1} = 3$

23. $\lim_{n\to\infty} \left(\sqrt{n+2} - \sqrt{n}\right) = 0$

24–69 Find the limit of the sequence if it converges, or prove that the sequence diverges. (You can use any theorem from this section.)

24. $a_n = \frac{2n-3}{n}$

25. $a_n = \frac{3n+5}{6n-1}$

26. $a_n = \frac{4n^2+1}{2n-1}$

27. $a_n = \frac{2n-1}{4n^2+1}$

28. $a_n = \frac{n+1}{\sqrt{n}}$

29. $a_n = (-1)^n \frac{\sqrt{n}}{n+1}$

30. $a_n = \frac{n \sin n}{n^2+2}$

31. $a_n = \sqrt{n+1} - \sqrt{n}$

32. $a_n = \frac{n^4 - 5n^3 + 2n^2 + 1}{3n^4 + n^2 + 2}$

33. $a_n = \frac{\pi^n}{n^2}$

34. $a_n = \frac{\pi^n}{2^n}$

35. $a_n = \frac{\pi^n}{4^n}$

36. $b_n = \frac{\ln n}{2n}$

37. $c_n = 0.5 + (-0.5)^n$

38. $a_n = 4^n + \left(\frac{1}{4}\right)^n$

39. $a_n = \frac{\ln \frac{1}{\sqrt{n}}}{\sqrt{n}}$

40. $d_n = \frac{n3^n}{4^n}$

41. $h_n = \frac{n^4 - 4}{e^n}$

42. $a_n = \frac{n!}{4^n}$

43. $a_n = \ln\left(3n^2+2\right) - \ln\left(n^2+7\right)$

44. $a_n = (-1)^n \frac{\sqrt{n}}{\sqrt{n}+1}$

45. $p_n = 2^{-1/n}$

46. $q_n = \sqrt{1 + \frac{1}{n}}$

47. $a_n = \sin^{-1}\left(\frac{n^2+n+2}{2n^2+1}\right)$

48. $a_n = \left(\frac{n^2+1}{3n^2-2}\right)\left(3 + \frac{1}{n}\right)$

49. $r_n = \frac{2}{(0.6)^n}$

50. $s_n = \frac{\ln 3n}{\ln 4n}$

51. $a_n = \left(\frac{1}{n}\right)^{1/n}$

52. $a_n = \frac{n^{1/n}}{\ln n}$

53. $t_n = \frac{n^n}{n!}$

54. $u_n = \tan^{-1}(\ln n)$

55. $a_n = \left(\frac{1}{\ln n}\right)^{1/n}$

56. $a_n = n \sin \frac{1}{n}$

57. $m_n = e^{-(\cos n)/n}$

58. $N_k = (-1)^k \frac{(\ln k)^2}{k}$

59. $a_n = \frac{2^n - 1}{\pi^n}$

60. $K_n = \sqrt[n]{3n+1}$

61. $L_n = \left(\frac{1}{10}\right)^{-1/n}$

62. $S_k = \left(\frac{1}{k}\right)^{2/k}$

63. $T_k = \frac{\left(\frac{1}{2}\right)^k}{k^k - 1}$

64. $a_n = \left(1 + \frac{1}{n}\right)^n$

65. $a_n = \left(1 + \frac{1}{n}\right)^{2n}$

66. $a_n = \left(1 + \frac{1}{n^2}\right)^n$

67. $a_n = \left(1 + \frac{3}{n}\right)^n$

68. $a_1 = 1, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{5}{a_n}\right)$

69. $a_1 = 1, \quad a_n = \frac{3}{4}a_{n-1} + \frac{1}{a_{n-1}}$

70–73 Use the Squeeze Theorem to prove that the given sequence converges.

70. $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$

71. $\left\{\frac{\cos^2 n}{3^n}\right\}_{n=1}^{\infty}$

72. $\left\{(-1)^n \frac{(\ln n)^2}{n^3}\right\}_{n=1}^{\infty}$

73. $\left\{2^{-n}\cos n\right\}_{n=1}^{\infty}$

74.* Prove the Bounded Monotonic Sequence Theorem. (**Hint:** Suppose first that $\{a_n\}$ is increasing and bounded above. Let L be the least upper bound of the set of values $\{a_n | n \in \mathbb{N}\}$ and fix an $\varepsilon > 0$. Since $L - \varepsilon$ is *not* an upper bound for $\{a_n | n \in \mathbb{N}\}$, there is an index N such that $a_N > L - \varepsilon$. Use monotonicity to finish the argument. Note that the decreasing case can be handled similarly, or by considering the sequence $\{-a_n\}$.)

75–80 Use the Bounded Monotonic Sequence Theorem to prove that the given sequence converges. In Exercises 75–78, find the limit.

75. $a_n = \frac{n^2}{n^2+1}$

76. $a_1 = \sqrt{6}, \quad a_{n+1} = \sqrt{6 + a_n}$

77. $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}$

78. $a_1 = 0, \quad a_n = \frac{1}{2 - a_{n-1}}$

79. $a_n = \frac{(1)(3)(5)\cdots(2n-1)}{(2)(4)(6)\cdots(2n)}$

80. $a_n = 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$

81–82 The Fibonacci sequence has many interesting applications in combinatorics and the mathematics of computer algorithms. Fibonacci numbers also appear in nature, in the arrangements of leaves and flower petals and the geometry of some shells. We can derive the explicit formula for F_n, the n^{th} term of the Fibonacci sequence, as follows. First, notice that φ and ψ in Example 2 are the roots of the quadratic equation $x^2 - x - 1 = 0$ and, thus, we have $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$. Using, say, $\varphi^2 = \varphi + 1$ we obtain the following equations.

$$\varphi^3 = \varphi\varphi^2 = \varphi(\varphi+1) = \varphi^2 + \varphi = (\varphi+1) + \varphi = 2\varphi + 1$$

$$\varphi^4 = \varphi\varphi^3 = \varphi(2\varphi+1) = 2\varphi^2 + \varphi = 2(\varphi+1) + \varphi = 3\varphi + 2$$

Notice that the coefficients are Fibonacci numbers, and if we repeat the process for higher powers of φ, the Fibonacci numbers keep coming up. More precisely, using an induction argument one can show the following relationships.

$$\varphi^n = F_n\varphi + F_{n-1} \quad (1)$$

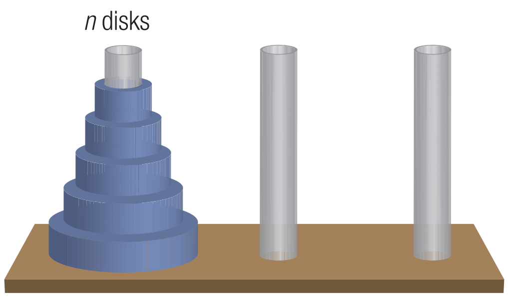
$$\psi^n = F_n\psi + F_{n-1} \quad (2)$$

In Exercises 81–82, use this observation to derive the explicit formula for F_n.

81. Verify the explicit formula given in Example 2 for the n^{th} term of the Fibonacci sequence. (**Hint:** Subtract equation (2) from (1) above and solve for F_n.)

82.* Complete the induction argument referred to in the discussion preceding Exercise 81 to prove $\varphi^n = F_n\varphi + F_{n-1}$.

83–84 The popular Tower of Hanoi puzzle was invented by the French mathematician Édouard Lucas in 1883. It consists of three pegs, with n disks placed on the first peg in order of increasing size from top to bottom (see figure). The objective is to transfer all disks to the second peg so that they end up in the same order, according to the following two rules. Only one disk can be moved at a time, and no disk can be placed at any time on top of a smaller disk. In Exercises 83–84, you will find T_n, the number of moves necessary to solve the puzzle with n disks.



83. Find a recursive definition for the sequence $\{T_n\}$. (**Hint:** Use T_{n-1} steps to move the top $n - 1$ disks to the third peg, then place the largest disk on the second peg, and then repeat the previous moves to complete the tower on the second peg.)

84.* Prove that an explicit formula for $\{T_n\}$ is $T_n = 2^n - 1$.

85.* Prove that if $a > 1$, then $\lim_{n\to\infty} a^n = \infty$.

86.* Prove that if $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$.

87.* Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ is convergent, then it is bounded.

88. Give an example of an unbounded sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \not\to \infty$ and $a_n \not\to -\infty$.

89.* Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ is convergent, then it has a largest or a smallest term.

90. Suppose that $\lim_{n\to\infty} a_n = \infty$ and $\{b_n\}$ is a sequence such that there is an N with $b_n \geq a_n$ for all $n \geq N$. Prove that $\lim_{n\to\infty} b_n = \infty$.

91.* Prove that the limit of a convergent sequence is unique, that is, no convergent sequence can have two different limits. (**Hint:** Create an indirect argument by first assuming that $\{a_n\}$ converges to both L_1 and L_2, $L_1 \neq L_2$, and then using the definition of convergence.)

92.* We call $\{a_n\}$ a *null sequence* if $\lim_{n\to\infty} a_n = 0$. Prove that if $\{a_n\}$ is a null sequence and $\{b_n\}$ is bounded, then $\{a_n \cdot b_n\}$ is a null sequence.

93. Suppose that all terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are nonzero and $\lim_{n\to\infty} |a_n| = \infty$. Prove that $\{1/a_n\}$ is a null sequence.

94.* Prove that if $\lim_{n\to\infty} a_n = \infty$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n + b_n) = \infty$.

95.* If every term of a sequence is positive, it is called a *positive sequence*. Prove that if $\{a_n\}$ is a positive sequence and $\lim_{n\to\infty} (a_{n+1}/a_n) = L > 1$, then $\lim_{n\to\infty} a_n = \infty$.

96.* Prove that if $\{a_n\}$ is a positive sequence and $\lim_{n\to\infty} (a_{n+1}/a_n) = L < 1$, then $\{a_n\}$ is a null sequence.

97.* Prove that if $\lim_{n\to\infty} (a_{n+1} - a_n) = L > 0$, then $\lim_{n\to\infty} a_n = \infty$.

98.* Suppose that for two convergent sequences, $\lim_{n\to\infty} a_n = L_a$, $\lim_{n\to\infty} b_n = L_b$, and that $L_a < L_b$. Prove that there is a positive integer N such that $a_n < b_n$ for all $n \geq N$.

99.* Suppose that for two convergent sequences, $\lim_{n\to\infty} a_n = L_a$, $\lim_{n\to\infty} b_n = L_b$, and that $a_n < b_n$ for all n. Prove that $L_a \leq L_b$. Can we conclude that $L_a < L_b$?

100. Prove that for any irrational number $r \in \mathbb{R}$ there is a strictly increasing sequence $\{r_n\}$ of rational numbers so that $r_n \to r$.

101. Let $\{a_n\}$ be a recursively defined sequence as follows: $a_1 = 1$, $a_2 = 2$, and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for $n \geq 2$. Prove that the sequence converges, find an explicit formula for its n^{th} term, and find its limit. (**Hint:** Start by finding the "gaps" between consecutive terms: $|a_{n+1} - a_n|$.)

102.* If $a > 0$, prove that the recursively defined sequence $a_1 = 1$, $a_{n+1} = (a_n^2 + a)/(2a_n)$ is convergent, and its limit is $\sqrt{a}$. (**Hint:** Think about Newton's method.)

Concept Check

103–114 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

103. If $\{a_n\}_{n=1}^{\infty}$ is convergent and its limit is L, then for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many terms of $\{a_n\}_{n=1}^{\infty}$.

104. If for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$, then $\{a_n\}_{n=1}^{\infty}$ is convergent.

105. If $\{a_n\}_{n=1}^{\infty}$ is convergent, then so is $\{|a_n|\}_{n=1}^{\infty}$.

106. If $\{|a_n|\}_{n=1}^{\infty}$ is convergent, then so is $\{a_n\}_{n=1}^{\infty}$.

107. If the first n terms of a convergent sequence are altered, the resulting sequence still converges to the same limit.

108. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_n^2 = L^2$.

109. If $\lim_{n\to\infty} a_n^2 = L$, then $\lim_{n\to\infty} a_n = \sqrt{L}$.

110. If $\lim_{n\to\infty} a_n^3 = L$, then $\lim_{n\to\infty} a_n = \sqrt[3]{L}$.

111. If $\lim_{n\to\infty} a_n = L$, and $L > 1$, then $\lim_{n\to\infty} a_n^n = \infty$.

112. If both $\{a_n\}$ and $\{b_n\}$ diverge, then $\{a_n + b_n\}$ diverges.

113. If $\{a_n + b_n\}$ converges and $\{a_n\}$ converges, then $\{b_n\}$ also converges.

114. If $\{a_n b_n\}$ converges and $\{a_n\}$ diverges, then $\{b_n\}$ also diverges.