

25. Repeat Exercise 23 for the right circular cone of radius  $R$  and height  $h$ .
26. If  $S$  is a surface such as in Exercise 22, use the Divergence Theorem to show that the volume of the solid  $D$  bounded by  $S$  can also be found as

$$V(D) = \iint_S x \, dy \, dz = \iint_S y \, dz \, dx = \iint_S z \, dx \, dy.$$

27. Let  $Q$  be a single point charge at the origin, as in our discussion that follows Example 2. Show that  $\nabla \cdot \mathbf{E} = 0$  for the electric field  $\mathbf{E}$  created by  $Q$ .
28. Mimic our discussion following Example 2 to prove the following, slightly more general version of Gauss' Law.

If  $S$  is a closed surface satisfying the hypotheses of the Divergence Theorem,  $k$  is a constant and  $\mathbf{F} = \frac{k\mathbf{r}}{|\mathbf{r}|^3}$  is an inverse square field, then the flux of  $\mathbf{F}$  over  $S$  is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \begin{cases} 4k\pi & \text{if } S \text{ encloses the origin.} \\ 0 & \text{otherwise} \end{cases}$$

- 29.\* Recall the uniformly charged sphere of radius  $R$  and total charge  $Q$  from Exercises 55–57 of Section 15.6. In this problem, you will provide a “second solution” to the aforementioned Exercise 57 as follows. Use Gauss' Law to show that the electric field  $\mathbf{E}$  due to the sphere is

$$\mathbf{E} = \begin{cases} \frac{\varepsilon Q \mathbf{e}_r}{r^2} & \text{if } R < r \\ 0 & \text{if } r < R. \end{cases}$$

(Recall  $\mathbf{e}_r$  from Exercise 54 of Section 15.1.)

**Hint:** Note that  $\mathbf{E}$  is a central force as in Exercise 29 of Section 15.7. Using the notation  $\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r$ , show that the flux across a sphere of radius  $r$  is  $4\pi r^2 E(r)$  and use Gauss' Law. Finally, note that if  $r < R$ , the total charge enclosed by any sphere of radius  $r$  is zero, and hence so is  $\mathbf{E}$ .)

30. Use the Divergence Theorem to provide a second solution to Exercise 28 of Section 15.7. (**Hint:** See Exercise 39 of Section 15.4.)
- 31.\* Prove that if  $\mathbf{F}$  has continuous partial derivatives, then its divergence at a point  $P$  can be obtained from the formula

$$\nabla \cdot \mathbf{F}(P) = \lim_{R \rightarrow 0} \frac{3}{4\pi R^3} \iint_{S_R(P)} \mathbf{F} \cdot \mathbf{n} \, d\sigma,$$

where  $S_R(P)$  is the sphere of radius  $R$  centered at  $P$ . (**Hint:** Use the Divergence Theorem; then argue that the divergence function “assumes its average value” somewhere in the closed ball bounded by  $S_R(P)$ . For a refresher on average value, see Section 5.2.)

**32–35** Prove the identity, assuming the scalar-valued functions  $f$  and  $g$  have continuous partial derivatives at least through the second order, and  $S$  and  $D$  satisfy the hypotheses of the Divergence Theorem. Note that  $\nabla^2 f$  stands for  $f_{xx} + f_{yy} + f_{zz}$ , i.e.,  $\nabla^2 f = 0$  means that  $f$  satisfies Laplace's equation (see Section 13.3). Recall also that  $D_n f$  stands for the directional derivative of  $f$  in the direction of  $\mathbf{n}$ . (For a review of directional derivatives, see Section 13.5.)

$$32.* \iint_S D_n f \, d\sigma = \iiint_D \nabla^2 f \, dV$$

$$33.* \iint_S f D_n g \, d\sigma = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV$$

(This is called *Green's first identity*. **Hint:** See Exercise 44 in Section 15.4.)

$$34.* \iint_S f D_n f \, d\sigma = \iiint_D |\nabla f|^2 \, dV, \text{ if } f \text{ satisfies Laplace's}$$

equation on  $D$ . (**Hint:** You may give a direct proof, or use Exercise 33.)

$$35.* \iint_S (f D_n g - g D_n f) \, d\sigma = \iiint_D (f \nabla^2 g - g \nabla^2 f) \, dV \text{ (This}$$

is called *Green's second identity*. **Hint:** Use Green's first identity from Exercise 33 twice.)

## Chapter 15

### Review Exercises

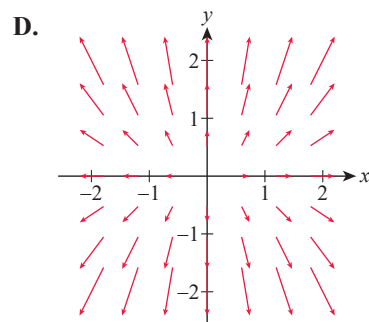
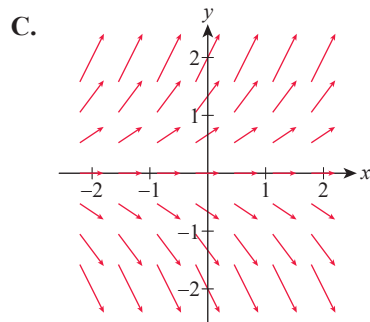
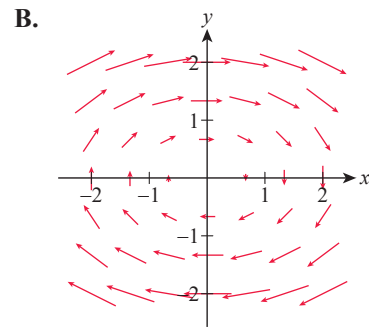
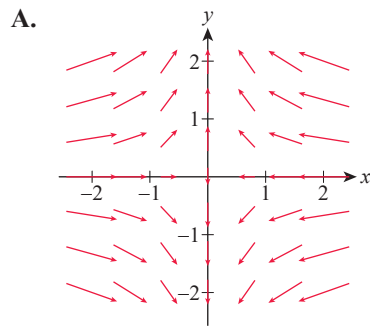
**1–4** Match the given planar vector field  $\mathbf{F}(x, y)$  with its graph (labeled A–D).

1.  $\mathbf{F}(x, y) = \langle 1, y \rangle$

2.  $\mathbf{F}(x, y) = \left\langle \frac{x}{2}, y \right\rangle$

3.  $\mathbf{F}(x, y) = \langle 2y, -x \rangle$

4.  $\mathbf{F}(x, y) = \left\langle -x, \frac{y}{\sqrt{|x, y|}} \right\rangle$



**5–6** Sketch the vector field  $\mathbf{F}$  by hand, using a few representative vectors.

5.  $\mathbf{F}(x, y) = \langle \sqrt{|x|}, y \rangle$

6.  $\mathbf{F}(x, y) = \langle \sqrt{x^2 + y^2}, \sqrt{x^2 + y^2} \rangle$

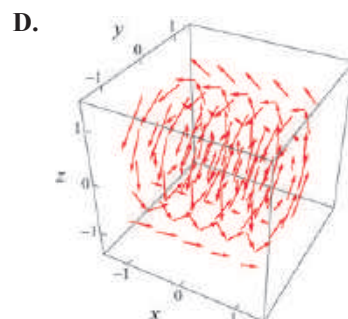
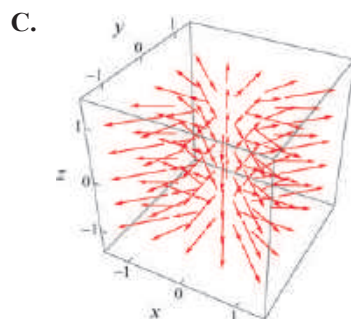
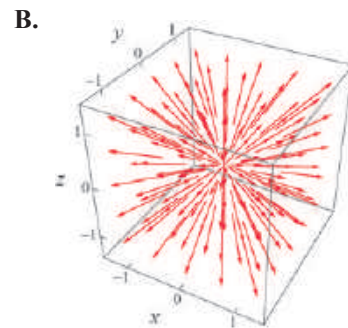
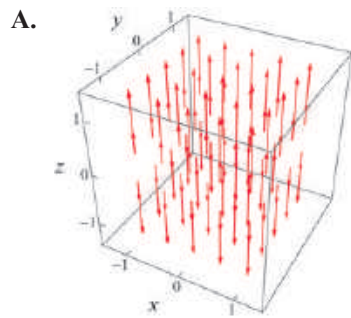
**7–10** Match the given three-dimensional vector field with its graph (labeled A–D).

7.  $\mathbf{F}(x, y, z) = \langle x, y, 0 \rangle$

8.  $\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle$

9.  $\mathbf{F}(x, y, z) = \langle 0, -z, y \rangle$

10.  $\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{2\sqrt{x^2 + y^2 + z^2}}$



**11–12** Determine  $\nabla f$  for the given scalar field  $f(x, y, z)$ .

11.  $f(x, y, z) = 2x^2y + xz^3$

12.  $f(x, y, z) = \ln \sqrt{x^2z^4 + 4y^2}$

**13–16** Decide whether the given vector field is conservative.

13.  $\mathbf{F}(x, y) = \left\langle \frac{y^2}{(x+y^2)^2}, \frac{-2xy}{(x+y^2)^2} \right\rangle$

14.  $\mathbf{F}(x, y) = \left\langle -x^2y, \frac{x}{y} \right\rangle$

15.  $\mathbf{F}(x, y, z) = \left\langle 3x^2yz + \frac{1}{yz}, x^3z - \frac{x}{y^2z}, x^3y - \frac{x}{yz^2} \right\rangle$

16.  $\mathbf{F}(x, y, z) = \left\langle \frac{3x^2y}{z}, \frac{x^3}{z}, \frac{x^3y}{z^2} \right\rangle$

**17–18** Find a potential function for the given conservative vector field.

17.  $\mathbf{F}(x, y) = \langle 12x^2 - 4xy, 3y^2 - 2x^2 \rangle$

18.  $\mathbf{F}(x, y) = \langle y^2 \cos(xy), xy \cos(xy) + \sin(xy) \rangle$

**19–22** Evaluate the line integral.

19.  $\int_C x^2y \, ds$ , where  $C$  is the semicircle  $y = \sqrt{4-x^2}$ , traversed counterclockwise

20.  $\int_C \left( x - \frac{y}{2} \right) ds$ , where  $C$  is the arc of the unit circle centered at the origin connecting  $(1, 0)$  with  $(0, 1)$ , followed by the line segment to the point  $(2, 2)$

21.  $\int_C (xz + 3y) \, ds$ , where  $C$  is the line segment joining  $(2, 0, 3)$  and  $(1, 3, 0)$

22.  $\int_C 2z(x-y) \, ds$ , where  $C$  is the helix  $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle$ , for  $0 \leq t \leq 6\pi$

23. Determine the center of mass of the V-shaped object of constant density connecting the point  $(-1, -2, 4)$  with the origin, and then the origin with the point  $(1, 8, 4)$ .

24. Suppose that in Exercise 23, the density of the object at any point is proportional to the distance from the  $xy$ -plane. Find its mass and center of mass in this case.

25. Determine the center of mass of the object with constant density that consists of a line segment connecting the point  $(2, -1, -\sqrt{2})$  with the point  $(1, 0, 0)$ , followed by the helix  $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$ ,  $0 \leq t \leq 1$ .

26.\* Find the mass and center of mass of the object in Exercise 25 if the density at any point is proportional to the distance from the  $xy$ -plane.

**27–28** Determine the work done by the force field  $\mathbf{F}$  in moving a particle along the indicated curve  $C$ .

27.  $\mathbf{F}(x, y, z) = \langle 2y^2, -xz, 3y \rangle$ , where  $C$  is defined by  $\mathbf{r}(t) = \langle t, 2t^2, 3t^3 \rangle$ ,  $0 \leq t \leq 2$

28.  $\mathbf{F}(x, y, z) = \langle -y, x, 3z \rangle$ , where  $C$  is the helix defined by  $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, t \rangle$ ,  $0 \leq t \leq 4\pi$

**29–30** A fluid's velocity field is given by  $\mathbf{F}(x, y, z)$ . Determine the fluid's flow along the indicated curve  $C$ .

29.  $\mathbf{F}(x, y, z) = \langle 2xyz, x^2z, x^2y \rangle$ , where  $C$  is the line segment connecting the points  $(0, 2, -1)$  and  $(1, 4, 5)$

30.  $\mathbf{F}(x, y, z) = \langle y^2, 2xy, 4z \rangle$ , where  $C$  is the curve defined by  $\mathbf{r}(t) = \langle \sin t, \cos t, t/4 \rangle$ ,  $0 \leq t \leq 3\pi$

31. Suppose the semicircle parametrized by  $\mathbf{r}(t) = \langle 3 \cos t, 0, 3 \sin t \rangle$ ,  $0 \leq t \leq \pi$  has charge density  $q(x, y, z) = \frac{2+z}{10^6}$  C/m. Find the electrostatic potential at the point  $(0, 4, 0)$ . (See Exercise 50 in Section 15.2.)

**32–35** Use the Fundamental Theorem for Line Integrals to evaluate the integral.

32.  $\int_C 2(x+y^2) \, dx + 4xy \, dy$ , where  $C$  is a piecewise smooth path from  $(0, 0)$  to  $(3, 2)$

33.  $\int_C e^{xy} (xy+1) \, dx + x^2 e^{xy} \, dy$ , where  $C$  is a piecewise smooth path from  $(0, -1)$  to  $(1, 1)$

34.  $\int_C (yz^2 + e^x \sin y) \, dx + (xz^2 + e^x \cos y) \, dy + 2xyz \, dz$ , where  $C$  is a piecewise smooth path from  $(0, -\pi/2, 2)$  to  $(0, 0, 0)$

35.  $\int_C \left( \frac{2x}{x^2+y^2} - 2y^3 \right) dx + \left( \frac{2y}{x^2+y^2} - 6xy^2 - z \sin y \right) dy + \cos y \, dz$ , where  $C$  is a piecewise smooth path from  $(2, 3, 0)$  to  $(1, 0, 2)$

**36–37** Show that the line integral is not path independent by finding two different values for the integral along two different paths connecting  $A$  and  $B$ . (Answers may vary.)

36.  $\int_C xy \, dx - 5y \, dy$ ;  $A(0, 0)$ ,  $B(1, 1)$

37.  $\int_C 2x \, dx + x \, dy + yz \, dz$ ;  $A(2, 0, 0)$ ,  $B(-2, 0, 0)$

**38–39** Show that the force field  $\mathbf{F}$  is conservative, and use this fact to determine the work done by  $\mathbf{F}$  in moving an object from  $A$  to  $B$ .

**38.**  $\mathbf{F}(x, y) = \langle \cos y + y \sin x, -\cos x - x \sin y \rangle$ ;  
 $A(0, 1), B(-1, -1)$

**39.**  $\mathbf{F}(x, y, z) = \left\langle \ln(y^2), \frac{2x}{y} + z^2, 2yz \right\rangle$ ;  
 $A(1, 1, 1), B(2, \sqrt{e}, 2)$

**40.** Use the Fundamental Theorem for Line Integrals to verify your answer for Exercise 29.

**41.** Use the Fundamental Theorem for Line Integrals to verify your answer for Exercise 30.

**42.** Recall the vector field  $\mathbf{F}(x, y) = \left\langle \frac{-y}{\sqrt{\langle x, y \rangle^2}}, \frac{x}{\sqrt{\langle x, y \rangle^2}} \right\rangle$   
 from Exercise 35 of Section 15.3. Show that  $\mathbf{F}$  is conservative on any region disjoint from the origin.

**43.** Show that the vector field

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{\sqrt{\langle x, y \rangle^p}}, \frac{x}{\sqrt{\langle x, y \rangle^p}} \right\rangle \text{ is not conservative if } p \neq 2.$$

**44–47** Find the divergence and curl of the vector field  $\mathbf{F}$ .

**44.**  $\mathbf{F}(x, y) = \langle x^2 + 4xy, 2x^2y \rangle$

**45.**  $\mathbf{F}(x, y) = \left\langle \frac{x}{y^2 + 1}, y(x^2 - x) \right\rangle$

**46.**  $\mathbf{F}(x, y, z) = \langle 2x(z^2 - 1), \ln(x^2 + y^2), x^2yz \rangle$

**47.**  $\mathbf{F}(x, y, z) = \langle x \sin(yz), y^2z, x(y + z) \rangle$

**48.** Consider the function  $f(\mathbf{r}) = \frac{1}{r^4}$ , where, as usual,  
 $r = |\mathbf{r}|$ . Prove:  $\nabla f = -\frac{4\mathbf{r}}{r^6}$ .

**49–50** Verify Green's Theorem by demonstrating the equality  $\oint_C P dx + Q dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$  for the given

$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  and region  $R$  enclosed by the piecewise smooth closed curve  $C$ .

**49.**  $P(x, y) = 2x + y^2, Q(x, y) = xy$ , where  $R$  is the triangle with vertices  $(-1, 0), (1, 0)$ , and  $(0, 1)$

**50.**  $P(x, y) = x - y, Q(x, y) = 2y$ , where  $R$  is the region bounded by the  $x$ -axis and the upper semicircle  $y = \sqrt{1 - x^2}$

**51–52** Use Green's Theorem to evaluate the indicated line integral.

**51.**  $\oint_C y^2 dx + 2xy dy$ , where  $C$  is the boundary of the region enclosed by the graphs of  $y = \sqrt{x}, x = 1$ , and the  $x$ -axis, oriented counterclockwise

**52.**  $\oint_C \frac{x^3 y^2}{2} dx - x^2 y dy$ , where  $C$  is the boundary of the region bounded by the coordinate axes and the line  $y = 2 - x$ , oriented clockwise

**53–54** Use Green's Theorem to find the work done as a particle moves counterclockwise around the closed path  $C$  in the force field  $\mathbf{F}$ .

**53.**  $\mathbf{F}(x, y) = \langle -3x^2y, (2x - y)y \rangle$ , where  $C$  is the boundary of the triangle with vertices  $(0, -1), (0, 1)$ , and  $(2, 0)$

**54.**  $\mathbf{F}(x, y) = \langle x^2 + 2y, \cos y + 3x \rangle$ , where  $C$  is the circle  $x^2 + (y - 1)^2 = 1$  (**Hint:** Use polar coordinates.)

**55–56** Evaluate the outward flux of the vector field  $\mathbf{F}(x, y)$  across the given path  $C$ .

**55.**  $\mathbf{F}(x, y) = \langle 2xy, y \rangle$ ,  $C$  is the boundary of the upper half disk bounded by  $y = \sqrt{4 - x^2}$  and the  $x$ -axis, oriented counterclockwise (**Hint:** Use polar coordinates.)

**56.**  $\mathbf{F}(x, y) = \langle 4x(x + y), x - y \rangle$ ,  $C$  is the boundary of the region between the graphs of  $y = x^2$  and  $x = y^2$ , oriented counterclockwise

**57.** Use the line integral method (Exercise 84, Section 15.4) to find the area of the heptagon with vertices  $(1, 0), (3.5, 0.5), (4, 2), (3, 4), (1, 5), (0, 3)$ , and  $(0, 1)$ .

**58.** Let  $f_1$  and  $f_2$  be differentiable single-variable functions, and  $C$  a piecewise smooth, simple closed curve. Prove  $\int_C f_1(x) dx + f_2(y) dy = 0$ . (**Hint:** Use Green's Theorem.)

**59–64** Match the parametric surface with its graph (labeled A–F).

$$59. \mathbf{r}(s, t) = \left\langle \frac{\cos t}{s}, \frac{\sin t}{s}, s \right\rangle,$$

$$0 \leq s \leq 4, \quad 0 \leq t \leq 2\pi$$

$$61. \mathbf{r}(s, t) = \langle 4s, 2s \cos t, 2s \sin t \rangle,$$

$$0 \leq s \leq 5, \quad 0 \leq t \leq 2\pi$$

$$63. \mathbf{r}(s, t) = \langle s \sin t \cos s, s \sin t \sin s, 2s \rangle,$$

$$0 \leq s \leq 9, \quad 0 \leq t \leq 2\pi$$

$$60. \mathbf{r}(s, t) = \left\langle s \sin t, s \cos t, \frac{3\sqrt{s}}{2} \right\rangle,$$

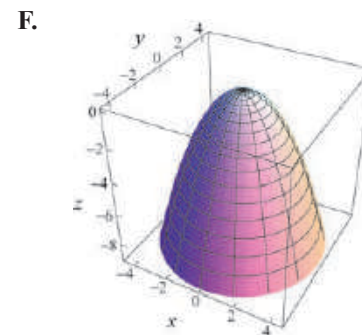
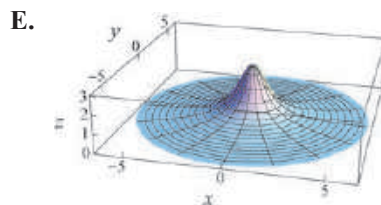
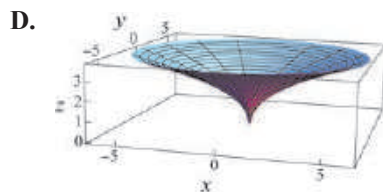
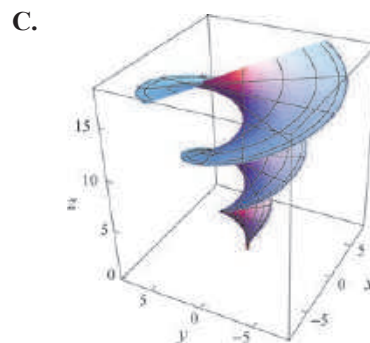
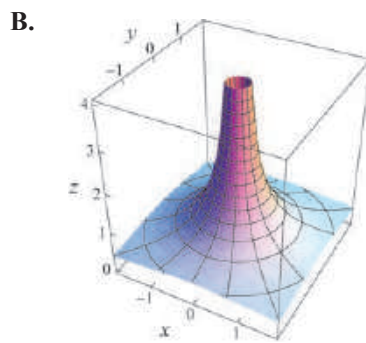
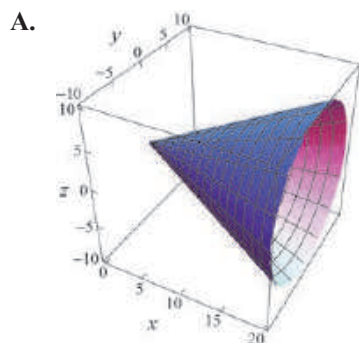
$$0 \leq s \leq 2\pi, \quad 0 \leq t \leq 2\pi$$

$$62. \mathbf{r}(s, t) = \left\langle \frac{3s}{2} \cos t, \frac{3s}{2} \sin t, -s^2 \right\rangle,$$

$$0 \leq s \leq 3, \quad 0 \leq t \leq 2\pi$$

$$64. \mathbf{r}(s, t) = \left\langle s \sin t, s \cos t, \frac{3}{1+s^2} \right\rangle,$$

$$0 \leq s \leq 2\pi, \quad 0 \leq t \leq 2\pi$$



**65–66** Describe the surface with a vector function of two parameters.

**65.** The graph of  $z = 1/y^2$ ,  $1 \leq y \leq 4$ , revolved about the  $y$ -axis

**66.** The graph of  $z = \tan x$ ,  $0 \leq x \leq \pi/4$ , revolved about the  $x$ -axis

**67–68** Identify the surface by examining its grid curves.

$$67. \mathbf{r}(s, t) = \langle 2s \sin t, s \cos t, s \rangle,$$

$$-\infty < s < \infty, \quad 0 \leq t \leq 2\pi$$

$$68. \mathbf{r}(s, t) = \langle \sin t, (2 + \cos t) \sin s, (2 + \cos t) \cos s \rangle,$$

$$0 \leq s \leq 2\pi, \quad 0 \leq t \leq 2\pi$$

**69–70** Obtain a parametrization for the indicated surface. (Answers will vary.)

$$69. z = 2x^2 + y^2$$

$$70. z = \sqrt{9 - 9x^2 - \frac{9}{4}y^2}$$

**71–72** Find an equation of the plane tangent to the surface at the indicated point.

$$71. \mathbf{r}(s, t) = \langle st, t - s, 2t^2 \rangle; \quad \mathbf{r}(1, 3)$$

$$72. \mathbf{r}(s, t) = \langle s \cos t, s \sin t, s^4 \rangle; \quad \mathbf{r}\left(1, \frac{\pi}{4}\right)$$

**73–76** Find the area of the surface  $S$ . (Use polar coordinates wherever they simplify your calculations.)

73.  $S$  is the surface common to the interior of the cylinder  $x^2 + y^2 = 9$  and the plane  $2z + 2y + x = 0$ .
74.  $S$  is the surface  $\mathbf{r}(s, t) = \langle s \cos t, s \sin t, 2s \rangle$  between the planes  $z = 2$  and  $z = 4$ .
75.  $S$  is the surface  $\mathbf{r}(s, t) = \left\langle s \cos t, s \sin t, \frac{s^2}{2} \right\rangle$  between the planes  $z = 1$  and  $z = 4$ .
76.  $S$  is the graph of  $4z = y^2 - x^2$  above the unit disk centered at the origin.

**77–78** A surface with variable mass density is given. Determine the mass of the surface.

77.  $\mathbf{r}(s, t) = \langle \cos t, \sin t, s \rangle$ ,  $0 \leq s \leq 1$ ,  $0 \leq t \leq \frac{\pi}{2}$ ; mass density is  $\rho(x, y, z) = z^2$
78.  $\mathbf{r}(s, t) = \langle 2 \cos t, 2 \sin t, s \rangle$ ,  $0 \leq s \leq 2$ ,  $0 \leq t \leq \pi$ ; mass density is  $\rho(x, y, z) = 8 - z^3$
79. Determine the center of mass of the surface of Exercise 77.
80. Determine the center of mass of the surface of Exercise 78.

**81–84** Determine the indicated flux of the vector field  $\mathbf{F}$  across the surface  $S$ . Unless otherwise specified, the surfaces are oriented with outward-pointing normal vectors.

81. The flux of  $\mathbf{F} = \langle 1, 2y, z \rangle$  out of the solid  $R$  bounded by  $z = \frac{x^2 + y^2}{2}$  and the plane  $z = 2$
82. The flux of  $\mathbf{F} = \langle 1, -z, y \rangle$  across the first-octant portion of the hemisphere of radius 2 centered at the origin
83. The flux of  $\mathbf{F} = \langle xy, 1, z \rangle$  across the first-octant portion of the cylinder  $x^2 + y^2 = 4$ ,  $0 \leq z \leq 6$
84. The flux of  $\mathbf{F} = \langle 2x, y, -z \rangle$  out of the solid  $R$  bounded by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$
85. Consider the thin conical surface of constant mass density, with both height and radius equaling  $R$ . Show that the second moment about its axis of rotation is  $I = mR^2/2$ , where  $m$  is the mass of the shell.

**86–87** Verify Stokes' Theorem by showing that the integrals

$\oint_C \mathbf{F} \cdot \mathbf{T} ds$  and  $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$  are equal.

86.  $\mathbf{F}(x, y, z) = \langle -4x - y, -2z, y \rangle$ , where  $S$  is the upper hemisphere of radius 3 centered at the origin
87.  $\mathbf{F}(x, y, z) = \langle 2y, -x, z^2 \rangle$ , where  $S$  is the surface  $z = 4 - x^2 - y^2$ ,  $z > 0$

**88–91** Use Stokes' Theorem to evaluate the indicated line integral.

88.  $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ , where  $\mathbf{F} = \langle 2x^2, -z, y^2 \rangle$  and  $C$  is the intersection of the cylinder  $x^2 + y^2 = 4$  and the plane  $2x + \frac{y}{2} + z = 6$ , with positive orientation when viewed from above
89.  $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ , where  $\mathbf{F} = \langle -2y^2, z^2, -x^2 \rangle$  and  $C$  is the intersection of the cylinder  $x^2 + y^2 = 2x$  and the plane  $z = \frac{x}{2}$ , with positive orientation when viewed from above
90.  $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ , where  $\mathbf{F} = \langle -y, -2z, -3x \rangle$  and  $C$  is the intersection of  $\frac{z}{2} = x^2 + y^2$  and  $z = 12 - x^2 - y^2$ , with positive orientation when viewed from above
91.  $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ , where  $\mathbf{F} = \langle -xz, 2z, x - y^2 \rangle$  and  $C$  is the intersection of  $z = x^2 + y^2$  and  $z = 4x + 5$ , with negative orientation when viewed from above

**92–95** Use Stokes' Theorem to evaluate the surface integral.

92.  $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$ , where  $\mathbf{F} = \langle -2z, x, 3xy \rangle$  and  $S$  is the hemisphere  $z = \sqrt{4 - x^2 - y^2}$ , oriented with an upward-pointing unit normal vector field
93.  $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$ , where  $\mathbf{F} = \langle 3y, z, -2x \rangle$  and  $S$  is the portion of the paraboloid  $z = x^2 + y^2$  with  $0 \leq z \leq 4$ , oriented with a downward-pointing unit normal vector field
94.  $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$ , where  $\mathbf{F} = \langle z^2, 2x^2, x^2 - y \rangle$  and  $S$  is the triangle determined by the first-octant portion of the plane  $x + 2y + 3z = 6$ , oriented with an upward-pointing unit normal vector field

95.  $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$ , where  $\mathbf{F} = \left\langle x - \frac{yz^2}{4}, 3x^2z, z - y^2 \right\rangle$  and  $S$  is the cone frustum  $z = 2\sqrt{x^2 + y^2}$  with  $2 \leq z \leq 6$ , oriented with an inward-pointing unit normal vector field

**96–97** Verify the Divergence Theorem by showing the equality of the integrals  $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$  and  $\iiint_D \nabla \cdot \mathbf{F} \, dV$  for the given vector field  $\mathbf{F}$  on the solid  $D$ .

96.  $\mathbf{F}(x, y, z) = \left\langle 3z, \frac{y}{4}, -2x \right\rangle$ , where  $D$  is the ball of radius 3 centered at the origin
97.  $\mathbf{F}(x, y, z) = \langle 3xy, -x, 2z \rangle$ , where  $D$  is the solid bounded by the paraboloid  $z = 1 - x^2 - y^2$  and the  $xy$ -plane

**98–103** Use the Divergence Theorem to find the flux of the vector field  $\mathbf{F}$  over the surface of the given solid  $D$ . Consider cylindrical or spherical coordinates where appropriate.

98.  $\mathbf{F}(x, y, z) = \langle -x^2y, yz, 5z \rangle$ , where  $D$  is the solid bounded by the parabolic cylinder  $z = 4 - y^2$ , the  $xy$ -plane, and the planes  $x = 0$  and  $x = 2$
99.  $\mathbf{F}(x, y, z) = \langle 5x(z - y), z^2 + \cos x, y \sin x \rangle$ , where  $D$  is the solid cylinder bounded by  $x^2 + y^2 = 9$ , the  $xy$ -plane, and the plane  $z = 3$
100.  $\mathbf{F}(x, y, z) = \left\langle 2xy, y - z, \frac{x^2y}{3} \right\rangle$ , where  $D$  is the tetrahedron with vertices at the origin,  $(6, 0, 0)$ ,  $(0, 3, 0)$ , and  $(0, 0, 4)$
101.  $\mathbf{F}(x, y, z) = \langle 2x + z, x - y^3, 2z \rangle$ , where  $D$  is the solid inside the cylinder  $x^2 + y^2 = 4$ , bounded by the  $xy$ -plane and  $z = 3 - y$
102.  $\mathbf{F}(x, y, z) = \langle -xy^2, 3yz^2, y^3 \rangle$ , where  $D$  is the solid cylindrical shell  $2 \leq x^2 + y^2 \leq 4$  between the planes  $z = 1$  and  $z = 6$
103.  $\mathbf{F}(x, y, z) = \left\langle \frac{yz^2}{2}, \frac{y^3}{8}, x(z - x) \right\rangle$ , where  $D$  is the portion of the solid cone  $z = \sqrt{x^2 + y^2}$  between the planes  $z = 2$  and  $z = 4$
104. Write a paragraph discussing the relationships and analogies between the Fundamental Theorem of Calculus, the Fundamental Theorem for Line Integrals, Green's and Stokes' Theorems, and the Divergence Theorem.

## Concept Check

**105–112** Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

105. If a vector field  $\mathbf{F}$  is constant, then  $\nabla \times \mathbf{F} = \mathbf{0}$ .
106. If a vector field  $\mathbf{F}$  is constant, then  $\nabla \cdot \mathbf{F} = 0$ .
107. A vector field is conservative if and only if its curl is zero.
108. The Fundamental Theorem for Line Integrals can only be used if the underlying vector field is conservative.
109. If a force field  $\mathbf{F}$  is conservative, then the work done by  $\mathbf{F}$  on a particle moving along a smooth path is zero.
110. Green's Theorem and Stokes' Theorem are unrelated.
111. When a charged particle moves along a piecewise smooth closed curve in an electric force field, the total work done by the force field is zero.
112. If the vector field  $\mathbf{F}$  has continuous partials in an open neighborhood of the closed, piecewise smooth surface  $S$ , then  $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0$ .

## Chapter 15 Technology Exercises

**113–116** Use a graphing utility to find the mass and center of mass of the solid with the given density.

113. The solid bounded by the cone  $z = \sqrt{4 - x^2 - y^2}$  and the  $xy$ -plane, with constant density  $\rho$
114. The solid bounded by the paraboloid  $z = 4 - x^2 - y^2$  and the  $xy$ -plane, with constant density  $\rho$
115. The solid of Exercise 114, with its density at any point being proportional to the distance from the  $xy$ -plane
116. The solid of Exercise 114, with its density at any point being proportional to the distance from the  $z$ -axis