Chapter 14
Review Exercises

1. Estimate $\iint_R x^2 y\,dA$ on the square $R = [0,3] \times [0,3]$ by using the Riemann sum approximation corresponding to $n = m = 3$, with each sample point (x^*_{ij}, y^*_{ij}) chosen to be the center point of the respective subsquare, $1 \le i \le 3$ and $1 \le j \le 3$. Then use repeated integration to find the true value of the integral.

2. Use symmetry and the properties of double integrals to evaluate $\iint_R (2-x)\,dA$ over the rectangle $R = \{(x,y) \mid -1 \le x \le 1, -2 \le y \le 3\}$. (Do not use repeated integration.)

3–4 Suppose $\iint_{R_1} f(x,y)\,dA = 6$, $\iint_{R_2} f(x,y)\,dA = -1$, $\iint_{R_1} g(x,y)\,dA = 2$, and $\iint_{R_1 \cup R_2} g(x,y)\,dA = 8$ over the disjoint rectangular regions R_1 and R_2. Use the properties of double integrals to evaluate the given integral.

3. $\iint_{R_1 \cup R_2} [2f(x,y) + 3g(x,y)]\,dA$

4. $\iint_{R_2} [3f(x,y) - g(x,y)]\,dA$

5–6 The given iterated integral represents the volume of a well-known solid. Use a formula from geometry to evaluate the integral.

5. $\int_0^3 \int_0^{1-x/3} (6 - 2x - 6y)\,dy\,dx$

6. $\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4 - x^2 - y^2}\,dx\,dy$

7–10 Evaluate the iterated integral.

7. $\int_0^{\pi/2} \int_0^2 x \cos y\,dx\,dy$

8. $\int_0^1 \int_0^1 \frac{y}{x^2+1}\,dy\,dx$

9. $\int_0^6 \int_{x/3}^{8-x} (xy - 1)\,dy\,dx$

10. $\int_0^1 \int_y^{\sqrt{y}} xe^y\,dx\,dy$

11–12 Rework the indicated problem by reversing the order of integration and verify that the answer does not change.

11. Exercise 9

12. Exercise 10

13–14 Evaluate the given double integral over the indicated region. Choose the most convenient order of integration.

13. $\iint_R (x^2 - y^2)\,dA$; R is the triangle with vertices $(0,0)$, $(2,4)$, and $(6,0)$.

14. $\iint_R 2xy\,dA$; R is the region bounded by $y = 3x - x^2$ and $y = x - 3$.

15–16 Evaluate the integral by reversing the order of integration. (Note that integrating in the given order would be impossible.)

15. $\int_0^1 \int_y^1 \sqrt[4]{1+x^2}\,dx\,dy$

16. $\int_0^1 \int_{\sqrt{y}}^1 e^{-x^3}\,dx\,dy$

17. By integrating the function $f(x,y) = 1$, find the area of the region R bounded by the circle $x^2 + y^2 = 1$ and the ellipse $4x^2 + y^2 = 4$.

18. Use the Domination Property to show that if $f(x,y)$ is integrable on the bounded region R, then

$$\left| \iint_R f(x,y)\,dA \right| \le \iint_R |f(x,y)|\,dA.$$

19–20 Find the average value of $f(x,y)$ over the region bounded by the graphs of the given equations.

19. $f(x,y) = x(x+y)$; $y = x^2$, $2y = 12 - x^2$

20. $f(x,y) = 4x^3 y$; $4y = x^3$, $y = x$, $y \ge 0$

21–22 Find the center of mass of the plane region of varying density that is bounded by the graphs of the given equations.

21. $y = 3 - \frac{x}{2}$, $x = 0$, $y = 0$; $\rho(x,y) = x + y^2$

22. $y = x$, $y = \sqrt{2-x}$, $y = 0$; $\rho(x,y) = x\sqrt{y}$

23–26 Determine the second moments I_x, I_y, and I_0 for the thin plate of constant density ρ modeled by the planar region R. Then find its corresponding radii of gyration.

23. R: The triangle bounded by the graph of $y = 4 - 2x$ and the coordinate axes

24. R: The square connecting the four points $(\pm a, 0)$ and $(0, \pm a)$

25. R: The first-quadrant region bounded by the graph of $y = 4 - x^2$ and the coordinate axes

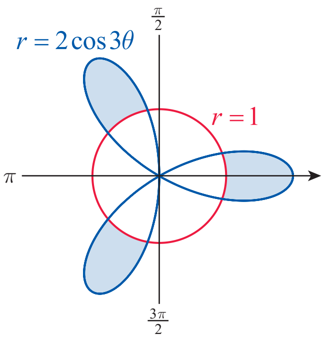
26.* R: The region bounded by the graph of $y = \dfrac{1-x^2}{1+x^2}$ and the x-axis

27. Suppose a thin plate of constant density ρ is modeled by the region R that is outside the square of diagonal length 2 centered at the origin, but inside the unit circle, also centered at the origin. Use the Principle of Superposition to find the moment of inertia of R about the origin.

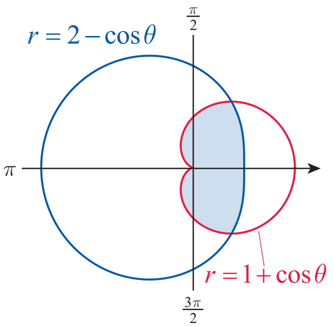
28. Suppose the thin plate of Exercise 25 has nonconstant density $\rho(x, y) = x$. Determine the second moments I_x, I_y, and I_0 and the corresponding radii of gyration.

29–30 Use a double integral in polar coordinates to find the area of the shaded region.

29.



30.



31–32 Use a double integral in polar coordinates to find the area of the given region R.

31. R: The region common to the circle $r = \frac{1}{2}$ and the cardioid $r = 1 - \cos\theta$

32. R: The region outside the circle $r = 1$ and inside the limaçon $r = \frac{1}{2} + \cos\theta$

33–34 Evaluate the double integral by changing to polar coordinates.

33. $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{dy\,dx}{\sqrt{x^2+y^2+2}}$

34. $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{\frac{x^2+y^2}{2}}\,dy\,dx$

35–36 Convert the integral into a Cartesian double integral and evaluate it.

35. $\displaystyle\int_0^{\pi/3} \int_0^{\sec\theta} r^2(\cos\theta - \sin\theta)\,dr\,d\theta$

36. $\displaystyle\int_{\pi/4}^{3\pi/4} \int_0^{4\csc\theta} r^4 \cos^2\theta \sin\theta\,dr\,d\theta$

37–38 Make your choice between the Cartesian and polar coordinate systems and evaluate the double integral.

37. $\displaystyle\iint_R \ln\sqrt{x^2+y^2}\,dA;\quad R\colon 1 \le x^2+y^2 \le e$

38. $\displaystyle\iint_R 14(x^2+y^2)^3\,dA;$ R is the region bounded by $y = 2x$, $y = 2$, and the y-axis.

39–42 Use double integration in polar coordinates on an appropriate region to find the volume of the solid S bounded by the given surfaces.

39. S: The solid bounded by the xy-plane and the paraboloid $z = x^2 + y^2 - 9$

40. S: The solid bounded by the xy-plane, the plane $z = 3x + 4y + 1$, and the cylinder $x^2 + y^2 = 16$

41. S: The solid common to the paraboloids $z = 18 - x^2 - y^2$ and $z = x^2 + y^2$

42. S: The solid common to the ellipsoid $8x^2 + 8y^2 + z^2 = 36$ and the cone $z = \sqrt{x^2+y^2}$

43–44 Evaluate the triple integral on the rectangular box S.

43. $\displaystyle\iiint_S (2x + yz)\,dV$, where $S = [-2,2]\times[1,4]\times[0,5]$

44. $\displaystyle\iiint_S xz^2\,dV$, where $S = [1,4]\times[-1,1]\times[2,3]$

45–46 Evaluate the iterated integral.

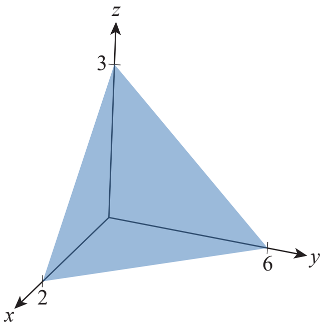
45. $\displaystyle\int_0^1 \int_0^{1-y} \int_0^{3-3x-3y} (2x - y)\,dz\,dx\,dy$

46. $\displaystyle\int_0^1 \int_0^{4-4x} \int_0^{2-\frac{y}{2}-2x} 5xy\,dz\,dy\,dx$

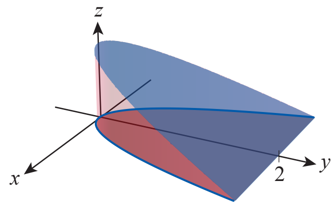
47–51 Use a triple integral to find the volume of the solid S.

47. S: The cylinder bounded by $y=\sqrt{1-x}$, the coordinate planes, and $z=4$. Use the following orders of integration: **a.** $dy\,dx\,dz$ **b.** $dz\,dy\,dx$

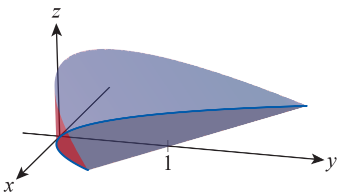
48. S: The tetrahedron bounded by the coordinate planes and $3x+y+2z=6$. Use the following orders of integration: **a.** $dx\,dy\,dz$ **b.** $dy\,dz\,dx$



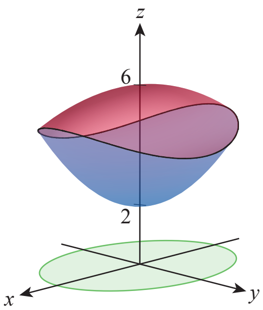
49. S: The solid bounded by the parabolic cylinder $y=2x^2$ and the planes $z=2-y$ and $z=0$. Use the following orders of integration:
a. $dz\,dy\,dx$ **b.** $dx\,dy\,dz$



50. S: The solid bounded by the parabolic cylinder $y=x^2/2$ and the planes $2x+4y+z=4$ and $z=0$. Choose the most convenient order of integration.



51. S: The solid bounded below by the surface $z=3x^2+y^2+2$ and above by the surface $z=6-x^2-y^2$. Choose the most convenient order of integration.



52–53 Revisit the indicated exercise, integrating in the given order. Verify that the answer does not change. (**Hint:** Sketching the solid of integration may be helpful.)

52. Exercise 45, order of integration: $dx\,dz\,dy$

53. Exercise 46, order of integration: $dy\,dz\,dx$

54–55 Use a triple integral to find the center of mass of the solid S with indicated density.

54. S: The solid of Exercise 49, with constant density

55. S: The tetrahedron bounded by the coordinate planes and $2x+y+z=3$, its density at each point being proportional to the distance from the yz-plane

56–57 Find the center of mass, second moments, and radii of gyration of the given solid S. Assume S is made of a substance with constant density ρ.

56. S: The first-octant region of the paraboloid $z=4-x^2-y^2$

57. S: The solid bounded by $z=6-x-3y$ and the coordinate planes

58. Write a short paragraph about when and why you would choose one particular coordinate system over the other two from among the Cartesian, cylindrical, and spherical systems.

59–60 Find a set of cylindrical coordinates for the point given in Cartesian coordinates.

59. $\left(\sqrt{2},-\sqrt{2},-\sqrt{2}\right)$ **60.** $\left(-1,-\sqrt{3},\sqrt{3}\right)$

61–62 Find a set of spherical coordinates for the point given in Cartesian coordinates.

61. $\left(\sqrt{2},\sqrt{2},-2\right)$ **62.** $\left(\sqrt{3},-3,-2\right)$

63–64 Find the Cartesian coordinates of the point given in cylindrical coordinates.

63. $\left(\sqrt{2},-\frac{\pi}{4},-\frac{\pi}{4}\right)$ **64.** $\left(-\pi,\frac{\pi}{2},1\right)$

65–66 Find the Cartesian coordinates of the point given in spherical coordinates.

65. $\left(4\sqrt{2},-\frac{\pi}{4},\frac{5\pi}{6}\right)$ **66.** $\left(4,\frac{\pi}{3},\frac{3\pi}{4}\right)$

67–68 Write the equation in cylindrical coordinates.

67. $x^2+(y-2)^2=z+4$ **68.** $2z(x^2+y^2)=y$

69–70 Write the equation in spherical coordinates.

69. $(x-1)^2+y^2+z^2=1$ **70.** $y=-\sqrt{3}x$

71–74 Change the given cylindrical or spherical equation into a rectangular one.

71. $r^2=1+z$

72. $r\csc\theta=3z$

73. $\varphi=\dfrac{2\pi}{3}$

74. $\sec\varphi+\rho=0$

75–76 Use cylindrical coordinates to determine the volume of the solid S.

75. S: The solid bounded by the paraboloid $z=9-(x^2+y^2)$ and the xy-plane

76. S: The solid bounded by the cylinder $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$, the cone $z=1-\sqrt{x^2+y^2}$, and the xy-plane

77–78 Use the cylindrical coordinate system to find the mass and the center of mass of the solid S with the given density function.

77. S: The solid of Exercise 75; $\rho(x,y,z)=9-z$

78. S: The solid of Exercise 76; $\rho(x,y,z)=z$

79–80 Use a triple integral in spherical coordinates to determine the volume of the solid S.

79. S: The solid bounded by the cone $z=\sqrt{\frac{1}{3}(x^2+y^2)}$ and the sphere $x^2+y^2+z^2=9$

80. S: The solid bounded by the sphere $x^2+y^2+z^2=1$ and the planes $y=x$ and $x=\sqrt{3}y,\ x>0$

81. Find the coordinates of the center of mass of the solid of Exercise 79, assuming constant density.

82. Find the mass of the solid of Exercise 80, assuming its density at any of its points is proportional to the distance from the origin.

83–84 Use a change of variables to evaluate the double integral on the given region R.

83. $\displaystyle\iint_R \frac{x+2y}{4x+3y}\,dA$; R is the region bounded by the coordinate axes and $x+2y=3$.

84. $\displaystyle\iint_R \frac{2x+5y}{x+4y}\,dA$; R is the region bounded by the coordinate axes and $2x+5y=6$.

85–86 Find a linear transformation $T(u,v)$ that maps a rectangular region onto the given parallelogram P.

85. P is bounded by $2y=x$, $2y=x+2$, $y=4x$, and $y=4x-12$.

86. P is bounded by $y=2x+1$, $y=2x+3$, $y=3-x$, and $y=-x-\frac{1}{2}$.

87. Evaluate $\displaystyle\iint_P (7x^2-14y)\,dA$ on the parallelogram P of Exercise 85.

88. Evaluate $\displaystyle\iint_P 3x^2y\,dA$ on the parallelogram P of Exercise 86.

89–90 Use a change of variables to evaluate the given integral on the solid R.

89. Evaluate $\displaystyle\iiint_R \frac{3y-x+2z}{2}\,dV$, where R is the solid bounded by the planes $x=0$, $x=4$, $z=0$, $z=2$, $3y=x$, and $3y=x+3$.

90. Evaluate $\displaystyle\iiint_R \left(\frac{y-3x}{3}+\frac{5z-2y}{4}\right)dV$, where R is the solid bounded by the planes $z=0$, $z=3$, $y=3x$, $y=3x+4$, and $z=2y$, $z=2y-6$.

91. Suppose $f(x,y,z)$ is defined on the box $R=[-a,a]\times[-b,b]\times[-c,c]$. Use the Fundamental Theorem of Calculus to prove the following formula.

$$\begin{aligned}\iiint_R f_{xyz}\,dV &= f(a,b,c)-f(a,b,-c)\\ &\quad -f(a,-b,c)+f(a,-b,-c)\\ &\quad -f(-a,b,c)+f(-a,b,-c)\\ &\quad +f(-a,-b,c)-f(-a,-b,-c)\end{aligned}$$

Concept Check

92–98 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

92. If $a>0$ and $f(x)$ is continuous, then $\int_0^a\int_0^y f(x,y)\,dx\,dy=\int_0^a\int_0^x f(x,y)\,dy\,dx$.

93. If $f(x, y) \neq 0$ on the bounded region R, then its average value on R cannot be zero.

94. The value of a double integral should be interpreted as the volume of the solid bounded by the graph of the integrand and one of the coordinate planes.

95. In order to find $\iint_R e^{\frac{x^2+y^2}{2}}\, dA$ on the rectangular region $R = \{(x, y) \mid |x| \leq a, |y| \leq b,\ a, b > 0\}$, the use of Cartesian coordinates is recommended.

96. In the cylindrical coordinate system, r can be negative.

97. In the spherical coordinate system, ρ can be negative.

98. If the binomial $x^2 + y^2$ is present in the integrand (or in the limits) of a triple integral, you should use cylindrical coordinates.

Chapter 14
Technology Exercises

99. Use a computer algebra system to find the center of mass of the solid of Exercise 80.

100. Write a program on your computer algebra system that performs a change from x- and y-coordinates to u- and v-coordinates and evaluates a given integral in the new coordinate system. Test your program by checking the answers you obtained for Exercises 89–90.

101. Use a computer algebra system to create the graph of the solid seen in Example 3 of Section 14.5.