# Chapter 13
## Review Exercises

**1–6** Match the function with its graph (labeled A–F).

**1.** $f(x,y)=\dfrac{-4}{2x^2+y^2+1}$

**2.** $f(x,y)=5\sqrt{1-\dfrac{x^2}{8}-\dfrac{y^2}{4}}$

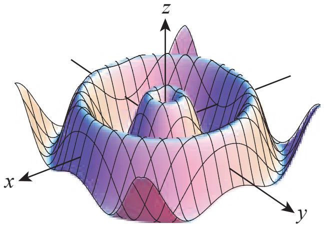
**3.** $f(x,y)=\dfrac{(x-2y)^2}{5}$

**4.** $f(x,y)=2.5\sin\sqrt{x^2+y^2}$

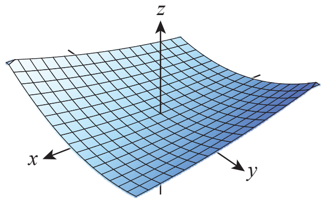
**5.** $f(x,y)=x^2-y^2$

**6.** $f(x,y)=\cos(2x-y)$

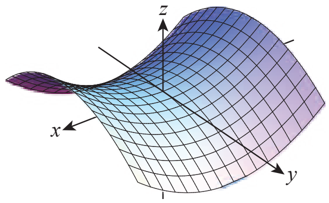
**A.**



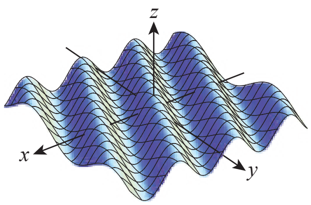

**B.**



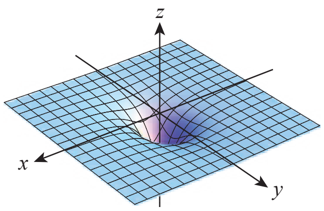

**C.**



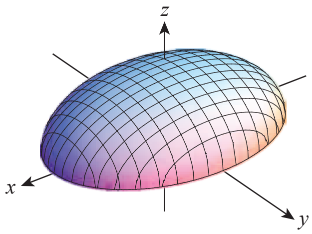

**D.**




**E.**

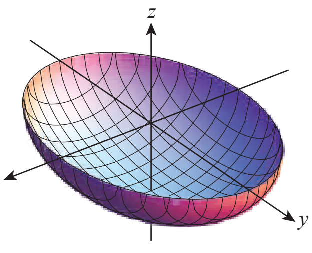



**F.**

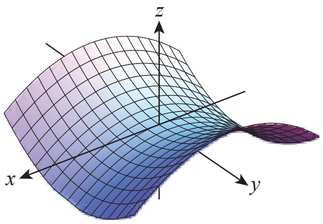



**7–12** Match the graph with its contour map (labeled A–F).

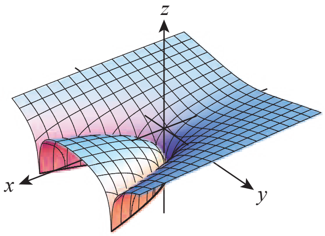
**7.**




**8.**




**9.**

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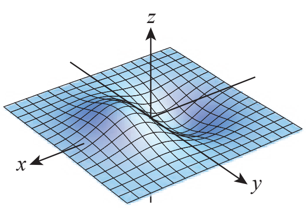
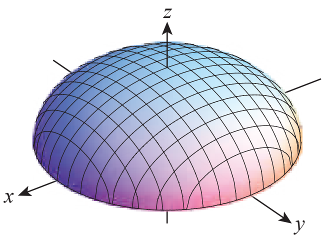
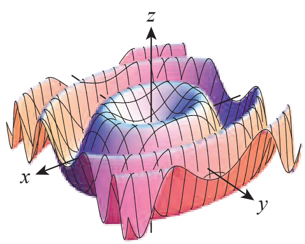


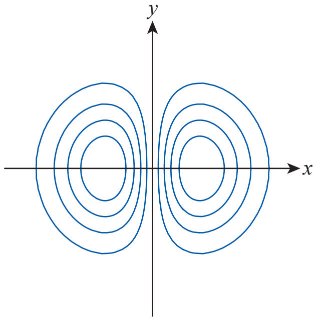

**11.**



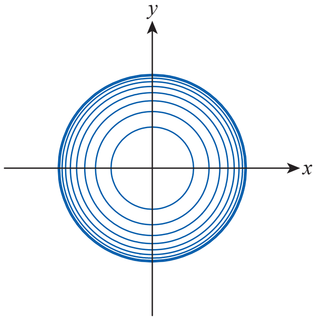

**12.**



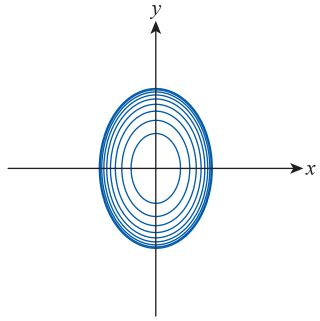
**A.**



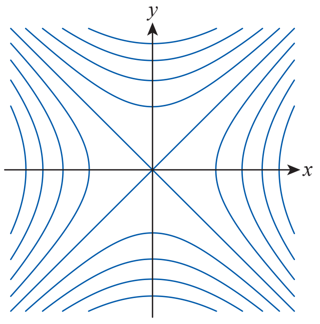

**B.**



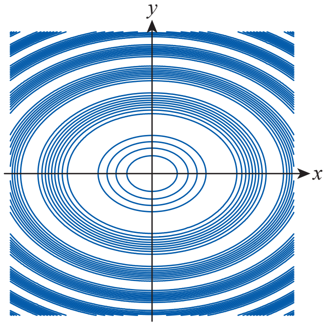

**C.**



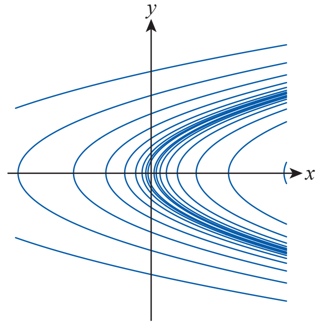

**D.**




**E.**




**F.**




**13–18** Determine whether the limit exists. If so, find it.

**13.** $\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x+y}$

**14.** $\lim_{(x,y)\to(1,1)} \frac{\ln(xy)}{\sin(\ln(xy))}$

**15.** $\lim_{(x,y)\to(0,1)} \frac{\sqrt{2x+y} - \sqrt{x+y}}{2x}$

**16.** $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\frac{\pi}{x^2+y^2}$

**17.** $\lim_{(x,y)\to(0,0)} \frac{\sin y}{xy^2}$

**18.** $\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^4}{x^2+y^4}$

**19.** Find two different paths in the $xy$-plane toward $(0,0)$ along which the limiting values of the function $f(x,y) = \frac{2xy^2}{x^3+y^3}$ are different. (Answers will vary.)

**20.** Use an $\varepsilon$-$\delta$ argument to show that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$

**21–24** Find all discontinuities (if any) of the given function. Classify them as removable or nonremovable.

**21.** $f(x,y) = \ln(x^4 + y^4)$

**22.** $g(x,y) = (6 - 2x^2 - 3y^2)^{-1}$

**23.** $h(x,y) = \arctan\frac{2}{(x^2-1)(y^2-4)}$

**24.** $k(x,y) = \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

**25–26** Determine $z_x$ and $z_y$ at the indicated point; then find the equations for the corresponding tangent lines that are parallel to, respectively, the $xz$-plane and the $yz$-plane.

**25.** $z = x^3y^2 + x^2y;\quad (-1,1)$

**26.** $z = (x^2y - 2)(y + 4x^2);\quad (1,2)$

**27–28** Find all first-order partial derivatives of the function.

**27.** $f(x,y,z) = y^2 \sin^{-1}(z\sqrt{x})$

**28.** $g(x,y,z) = \frac{ze^{xz}}{\sqrt{x+y^2}}$

**29–30** Use implicit differentiation to determine $z_x$ and $z_y$.

**29.** $2y^3 - x^2z - xyz^3 = 0$

**30.** $(z-y)^3 + 2\ln(xy) = z$

**31–32** Verify the equality of the mixed partials $f_{xy}$ and $f_{yx}$.

**31.** $f(x,y) = (x^3+y)^2 \ln(xy)$

**32.** $f(x,y) = \tan^{-1}\sqrt{x^2+y^2}$

**33–34** Verify that the third-order mixed partials $f_{xyz}$, $f_{yzx}$, and $f_{zxy}$ are equal.

**33.** $f(x,y,z) = 2x^2z^2 + xyz - 4zy^3$

**34.** $f(x,y,z) = xy\ln(z-2x)^2$

**35–36** Use the definition of the partial derivative to find $\partial f/\partial x$ and $\partial f/\partial y$ for the given function.

**35.** $f(x,y)=\sqrt{x}y^2$ **36.** $f(x,y)=\dfrac{x^3+y^2}{x}$

**37–38** If possible, find a function $f(x,y)$ that has the indicated partial derivatives. If such a function doesn't exist, explain why.

**37.** $f_x(x,y)=y^3+\dfrac{1}{x},\quad f_y(x,y)=3xy^2$

**38.** $f_x(x,y)=yx^2-\cos y,\quad f_y(x,y)=\dfrac{x^3}{3}-x\sin y$

**39–40** Determine whether the function is differentiable at the origin. Give a reason for your answer.

**39.** $f(x,y)=x\sin^2 y+xy^2$

**40.** $f(x,y)=\sqrt{x^2+y^2}\arctan(xy)$

**41.** Show that the piecewise–defined function

$$f(x,y)=\begin{cases}\dfrac{2xy}{\sqrt{x^6+y^6}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

is not differentiable at the origin, even though both $f_x(0,0)$ and $f_y(0,0)$ exist.

**42.** Use the definition to show that the function $f(x,y)=2x-3y^2$ is differentiable.

**43–46** Use the Chain Rule to determine $dy/dx$.

**43.** $y=2uv^3;\quad u=x^2-2,\quad v=3x$

**44.** $y=u^2e^{u^2v};\quad u=\sqrt{x},\quad v=x^2$

**45.** $y=u^2v\cos w;\quad u=\sqrt{x},\quad v=x^3,\quad w=x^2$

**46.** $y=uv^2+\sqrt{w};\quad u=\ln x,\quad v=2x,\quad w=x^2$

**47–49** Use the Chain Rule to find $f'(t)$ at the given point.

**47.** $f(x,y)=(x-2y)^2$ along the curve $x=t\sin t$, $y=2t\cos t$, at $t=\pi/2$

**48.** $f(x,y)=\ln(xy)$ along the curve $x=t^2$, $y=t-1$, at $t=2$

**49.** $f(x,y,z)=xyz$ along the space curve $x=t\cos t$, $y=t\sin t$, $z=2t$, at $t=\pi$

**50–54** Use the Chain Rule to determine the partial derivatives of the given function $f$. (Answers may be left in terms of the intermediate and independent variables.)

**50.** $f(u,v)=3u^2-v\sqrt{u};\quad u(x,y)=x^2,$ $v(x,y)=xy^2$

**51.** $f(u,v)=\ln(u^2+v^2);\quad u(x,y)=x\sqrt{y},$ $v(x,y)=x^2y$

**52.** $f(u,v,w)=u\cos v-\sin w;$ $u(x,y)=2x+y,\quad v(x,y)=y-x^2,$ $w(x,y)=xy$

**53.** $f(u,v,w)=u(v-w^2);\quad u(x,y)=2x^3,$ $v(x,y)=x-y^2,\quad w(x,y)=3x+y$

**54.** $f(u,v,w)=uve^w;\quad u(x,y,z)=yz^2,$ $v(x,y,z)=(x+y)^2,\quad w(x,y,z)=x-4z$

**55–56** Find $dy/dx$ where $y$ is given implicitly by the given equation. (**Hint:** Rewrite the equation in the form $F(x,y)=0$, and use the appropriate formula from Section 13.4.)

**55.** $x^2(2-xy)=xy^3$ **56.** $(x^3-y^2)^2=4x^2y$

**57–58** Find $z_x$ and $z_y$ where $z$ is defined implicitly by the given equation. (**Hint:** Use the appropriate formula from Section 13.4.)

**57.** $xy^2+yz^3=x^3z^2$ **58.** $ze^x+2x\sin y+xyz^3=1$

**59.** Suppose $z=f(x,y)$ has continuous second partials, $x=r\cos\theta$ and $y=r\sin\theta$. Use the Chain Rule to prove the following.

$$\frac{\partial^2 f}{\partial r^2}=\frac{\partial^2 f}{\partial x^2}\cos^2\theta+2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin^2\theta$$

**60.** The height of a right rectangular pyramid is increasing at a rate of 0.2 in./min, while the edge of its square base is decreasing at 0.1 in./min. Find the rate of change of its volume and surface area when the edge of the base is 15 in. long and the height is 20 in.

**61–64** Find the gradient of the function at the indicated point.

**61.** $f(x,y)=3x^2y-2\sqrt{x};\quad (1,3)$

**62.** $f(x,y)=\dfrac{\cos(x-y)}{y^2};\quad (1,1)$

**63.** $f(x,y,z)=x^2e^{yz};\quad (-1,0,2)$

**64.** $f(x,y,z)=\dfrac{\tan^{-1}(x+y)}{z};\quad (3,-2,4)$

**65–66** Use the definition to find the directional derivative of $f$ at the given point in the indicated direction.

**65.** $f(x,y,z)=\dfrac{xy^2}{z}$; $(2,4,1)$; $\mathbf{v}=\langle 3,1,2\rangle$

**66.** $f(x,y)=x+2\sqrt{y}$; $(1,1)$; $\mathbf{v}=\langle 1,2\rangle$

**67–70** Find the derivative of the function at the given point in the indicated direction.

**67.** $f(x,y)=\ln\sqrt{x^2+y}$; $(1,0)$; $\mathbf{v}=\langle -2,14\rangle$

**68.** $f(x,y)=x^2+3xy-2y^2$; $(1,2)$; in the direction toward $(-2,3)$

**69.** $f(x,y,z)=z\sqrt{\dfrac{xz+y}{x}}$; $(1,-1,2)$; in the direction toward $(4,-6,3)$

**70.** $f(x,y,z)=xy^2z^3$; $(1,1,1)$; $\mathbf{v}=\langle 3,2,1\rangle$

**71–72** Find the direction and value of the greatest rate of increase for the function at the given point.

**71.** $f(x,y,z)=\dfrac{y+z^2}{x+1}$; $(0,1,-1)$

**72.** $f(x,y)=4x^2+xy+\dfrac{x}{y}$; $(0,1)$

**73.** Suppose that the temperature in three-dimensional space around a heat source that is located at the origin can be described by the function

$$T(x,y,z)=450e^{-(2x^2+y^2+z^2)/2}.$$

**a.** Find the direction and magnitude of the temperature's maximum rate of decrease at the point $(1,1,1)$.

**b.** Find the rate of change of temperature at the point $(1,1,1)$ in the direction of the point $(1,2,3)$.

**74–75** Find the direction of no change for the function at the given point.

**74.** $f(x,y)=\dfrac{2xy}{x^2-y^2}$; $(2,-1)$

**75.** $f(x,y)=x\ln(y-x^2)$; $(1,2)$

**76–79** In Exercises 76–77, use the technique seen in this chapter to find an equation for the tangent line to the graph of the equation at the indicated point. In Exercises 78–79, generalize to obtain the respective tangent plane.

**76.** $2x^2+xy-3y^2+13=0$; $(2,3)$

**77.** $x^2(y+2)=4x^3$; $(1,2)$

**78.** $2y^2-x^2=2z^2$; $(4,3,1)$

**79.** $4z-3x^2=3y^2$; $(4,2,15)$

**80–81** Find a function with the given gradient. If it is not possible, explain why. (Answers will vary.)

**80.** $\nabla f=\left\langle \dfrac{1}{\sqrt{x}}+y^2, 2xy\right\rangle$ **81.** $\nabla f=\langle xy+1, x^2\rangle$

**82–85** Determine the tangent plane and the normal line at the indicated point of the given surface.

**82.** $5x^2-y^2-2z=0$; $(1,3,-2)$

**83.** $(2x-1)(5x+y^2)=z^2$; $(1,2,3)$

**84.** $\dfrac{x}{\sqrt{y^2+x}}-z=0$; $\left(3,1,\dfrac{3}{2}\right)$

**85.** $\ln(x^2+y^2+z^2)=0$; $\left(\dfrac{\sqrt{3}}{2},\dfrac{1}{2},0\right)$

**86–87** Find parametric equations for the line tangent to the intersection of the two surfaces at the indicated point.

**86.** $x^2+4y^2=6-z$; $2z=x^2+y^2$; $(1,1,1)$

**87.** $x^3-2y^3+xy^2-z^2=5xy$; $2y^2-x^2=z$; $(-1,1,1)$

**88–89** Find **a.** the differential $df$ at an arbitrary point $(x,y)$ and **b.** the linearization of $f$ at the indicated point.

**88.** $f(x,y)=(x-1)^3+y^2-x$; $(2,1)$

**89.** $f(x,y)=\sin(x^2-y^2)$; $(-1,1)$

**90–91** Find the linear approximation of the function at the indicated point.

**90.** $f(x,y,z)=x\tan^{-1}(yz)$; $(2,1,1)$

**91.** $f(w,x,y,z)=\dfrac{x^2y-wz^2}{xyw}$; $(1,2,-1,4)$

**92.** Suppose the base lengths of the rectangular pyramid of Exercise 60 were measured within a relative error of 2.5%, and its height within an error of 2%. Use differentials to find an upper bound on the relative error with which the pyramid's volume is known.

**93–96** Find any extrema of the given function. Use the Second Partial Derivative Test (if necessary) to classify critical points.

**93.** $f(x,y)=x^2+y^2-4x+2y+6$

**94.** $f(x,y) = 4y - y^2 - 8x - x^2$

**95.** $f(x,y) = x^3 + y^3 - 4y^2 - x^2$

**96.** $f(x,y) = 2y^4 - y^2 + x^2 + x$

**97–98** Find the absolute extrema of the given function on the indicated region.

**97.** $f(x,y) = x^2 - xy + 1;$
$D = \{(x,y) | -2 \le x \le 2, 0 \le y \le 2\}$

**98.** $f(x,y) = 4xy - 3x - y + 2;$ $D$: The triangle with vertices $(0,0)$, $(5,0)$, and $(0,4)$

**99.** Determine $m$ and $b$ such that the sum of the squares of vertical distances from the line $y = mx + b$ to the points $(1,2)$, $(4,2)$, $(6,6)$, and $(8,4)$ is minimal.

**100.** Find the minimum distance between the point $(2,-1,1)$ and the plane $z = 4x - y - 2$.

**101.** Find the minimum distance from the origin to the surface $z = 2xy - 1$.

**102.** A rectangular box is placed in the three-dimensional coordinate system with one vertex at the origin and the three edges containing it lying along the positive coordinate axes. If the opposite vertex lies in the plane $x + 4y + 2z = 6$, what is the greatest possible volume for such a box?

**103.** Suppose the sum of three positive numbers is 150, what is the greatest possible value for the sum of their squares?

**104–107** Find any extreme values of the function subject to the given constraint.

**104.** $f(x,y) = x^2 y;$ constraint: $\frac{x^2}{2} + \frac{y^2}{4} = 1$

**105.** $f(x,y) = x^2 + y^2;$ constraint: $xy = 4$

**106.** $f(x,y) = x^2 - y^2;$ constraint: $3x + 4y = 2$

**107.** $f(x,y) = x^2 + xy - y^2;$ constraint: $x = y^2$

**108–109** Use Lagrange multipliers to find the coordinates of the point $Q$ on the given surface that is closest to the point $P$.

**108.** $3z - 2x + y + 6 = 0;$ $P(1,0,1)$

**109.** $z = x + y^2;$ $P(1,0,0)$

**110–111** Find the absolute extreme value(s) of the function subject to the given constraints.

**110.** $f(x,y,z) = x - 2y + z;$
constraints: $x^2 + z^2 = 1$ and $x + y + z = 2$

**111.** $f(x,y,z) = 2(x^2 + y^2 + z^2);$
constraints: $x + y + z = 2$ and $2x + z = 1$

**112.** A small company profits \$10 per unit on products $A$ and $C$, while the profit on product $B$ is \$20 per unit. Let $x$, $y$, and $z$ denote hundreds of units produced from each product.

**a.** Find the profit function $P(x,y,z)$.

**b.** Assuming that the manufacturing process is under the constraint $x^2 + 10y^2 + 2z^2 \le 60$, find the maximum profit for the company under these conditions.

**113.** Find the points closest to and farthest from the origin on the rotated ellipse $2x^2 + xy + 2y^2 = 4$.

**114.** Find the highest and lowest points on the curve of intersection of the two surfaces $x^2 + 2y^2 + z^2 = 10$ and $x + 4y + z = 8$.

## Concept Check

**115–126** Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

**115.** Any vertical line (i.e., one that is parallel to the $z$-axis) can intersect the graph of a two-variable function $f(x,y)$ at most once.

**116.** If a function has three or more independent variables, then it has no graph.

**117.** If $f(x,y)$ is continuous at $(a,b)$ and $f(a,b) = L$, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

**118.** If $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ and $a \in \mathbb{R}$, then $\lim_{x\to 0} f(x,ax) = 0$.

**119.** If $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, then for any $\mathbb{R} > 0$ there exists $R \in \mathbb{R}$ such that $|f(x,y)| < \varepsilon$ whenever $0 < x^2 + y^2 < R$.

**120.** If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then $f(x,y) \to L$ as $(x,y)$ approaches $(a,b)$ along any curve in the $xy$-plane.

**121.** If $f(x,y)$ is continuous at $(a,b)$, then both $f_x(a,b)$ and $f_y(a,b)$ exist.

**122.** If $f(x,y)=\dfrac{g(x)}{h(y)}$, then

$$f_x(x,y)+f_y(x,y)=\frac{g'(x)h(y)-g(x)h'(y)}{h^2(y)}.$$

**123.** If $f_x(a,b)$ and $f_y(a,b)$ both exist, then $f(x,y)$ is differentiable at $(a,b)$.

**124.** If $f_x(x,y)$ and $f_y(x,y)$ are continuous at $(a,b)$, then $f(x,y)$ is differentiable at $(a,b)$.

**125.** If $f$ has a relative maximum or minimum at $(a,b)$, then all directional derivatives $D_{\mathbf{u}}f(a,b)$ are 0.

**126.** If $f$ has a relative maximum or minimum at $(a,b)$, then the tangent plane to its graph at $(a,b)$ is horizontal.

# Chapter 13
# Technology Exercises

**127–130.** Write a program on your computer algebra system or programmable calculator that accepts the equation of a surface and a point on the surface, and generates the corresponding tangent plane and normal line to the surface. Test your program by checking the answers you obtained for Exercises 82–85.

**131–134.** Write a program on your computer algebra system or programmable calculator that accepts a two-variable function $f(x,y)$ and then locates and classifies its critical points by utilizing the Second Partial Derivative Test. Use your program to check the answers you obtained for Exercises 93–96.

**135.** Recall the Ideal Gas Law, $pV=nRT$ from Section 3.4, Exercise 97. Suppose $\frac{1}{2}$ mole of an ideal gas is present in an expandable container. Assuming the volume of the container can expand up to 1 $\text{m}^3$, and that it can withstand a maximum pressure of 120 kPa along with temperatures of up to 500 K, use a computer algebra system to generate a contour map that illustrates how the volume depends on pressure when temperature is kept at constant levels. (Note that the resulting contour lines are called *isotherms*. **Hint:** Start by expressing $T$ as a function of $p$ and $V$ from the Ideal Gas Law.)

**136.** Mimic Exercise 135 to generate a few contour lines of the pressure function under the same conditions (these curves are called *isobars*).