

Chapter 12

Review Exercises

1–8 The following vector functions are defined on the interval $(0, 6\pi)$. Match each one with its graph (labeled A–H).

1. $\mathbf{r}(t) = \langle \cos t, 2\sin t, \sqrt{t} - 2 \rangle$

3. $\mathbf{r}(t) = \langle 1.5\sin t, \sqrt{t}, 1.5\cos t \rangle$

5. $\mathbf{r}(t) = \langle 2\cos t, \sqrt{t}, 2\sin^2 t \rangle$

7. $\mathbf{r}(t) = \langle 2.5\sin 12t, 3.5\cos t, 3.5\sin t \rangle$

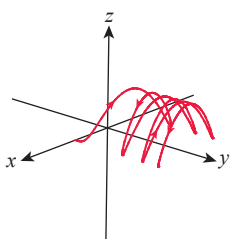
2. $\mathbf{r}(t) = \langle \frac{t}{2}\sin 3t, \frac{t}{2}\cos 3t, \frac{t}{2} \rangle$

4. $\mathbf{r}(t) = \langle \frac{t}{3}, 2\sin t, \frac{t}{3} \rangle$

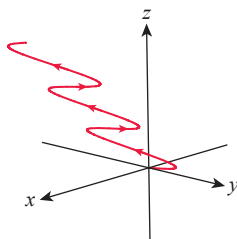
6. $\mathbf{r}(t) = \langle 0.1t\cos t, 0.2t\sin t, e^{0.1t} \rangle$

8. $\mathbf{r}(t) = \langle \frac{7\cos 5t}{2t}, \frac{7\sin 5t}{2t}, \frac{t}{5} \rangle$

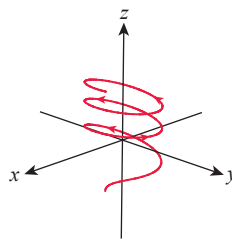
A.



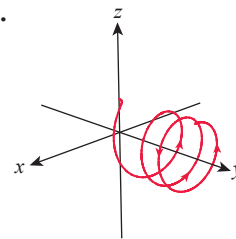
B.



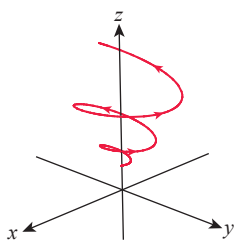
C.



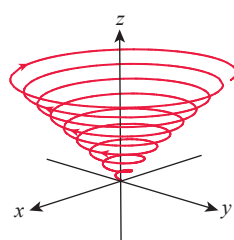
D.



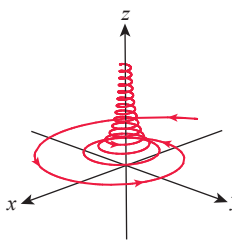
E.



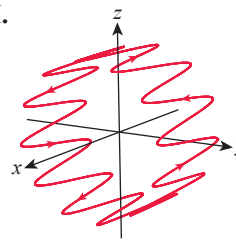
F.



G.



H.



9–10 Describe the intersection of the surfaces as a vector function. (Use the suggested parameter.)

9. The elliptic cylinder $5x^2 + 2y^2 = 10$ and the hyperbolic paraboloid $5x^2 - 2y^2 = z$ ($x = \sqrt{2}\cos t$)

10. The elliptic paraboloid $x^2 + 2y^2 = z$ and the parabolic cylinder $y = x^2$ ($x = t$)

11–12 Determine whether the indicated limit exists. If so, find it.

11. $\lim_{t \rightarrow 1} \left\langle \ln t, \sec(t-1), \frac{t-1}{t^2-1} \right\rangle$

12. $\lim_{t \rightarrow 0} \left\langle \frac{t}{t^2+1} \mathbf{i} + \arctan t \mathbf{j} + \frac{t^2}{|t|} \mathbf{k} \right\rangle$

13–14 Find any discontinuities of the given vector function.

13. $\mathbf{r}(t) = \left\langle e^{t+1}, \frac{t+1}{t^2+1}, \frac{t+1}{t^2-1} \right\rangle$

14. $\mathbf{r}(t) = |t-1|\mathbf{i} + \tan t \mathbf{j} - \arctan t \mathbf{k}$

15–16 Find $\mathbf{r}'(t)$.

15. $\mathbf{r}(t) = \langle \sec t, \arcsin \sqrt{t}, \tan^2 t \rangle$

16. $\mathbf{r}(t) = \cos 2t \mathbf{i} - \frac{t+2}{\sqrt{t-1}} \mathbf{j} + \ln(t^3-3) \mathbf{k}$

17–20 Find the vector form of an equation for the line tangent to the curve at the specified value of t .

17. $\mathbf{r}(t) = \langle 2\sin 4t, \cos t, 1-t^2 \rangle$; $t = 0$

18. $\mathbf{r}(t) = \langle \tan^{-1} 2t, t^{3/2}, 3t-2 \rangle$; $t = \frac{1}{2}$

19. $\mathbf{r}(t) = t\mathbf{i} - \ln(2t^2 + 1)\mathbf{j} - \sqrt{3t}\mathbf{k}; \quad t = 3$

20. $\mathbf{r}(t) = e^{t^2+2}\mathbf{i} + e^{\sqrt{t+1}}\mathbf{j} + e^{\cos t}\mathbf{k}; \quad t = 0$

21–24 Perform the integration—definite or indefinite, as indicated.

21. $\int \left\langle e^{t+1}, \frac{t}{\sqrt{t^2+2}}, \sin 2t \right\rangle dt$

22. $\int \left\langle \sqrt{t}\mathbf{i} - \frac{1}{\sqrt{1-t^2}}\mathbf{j} - t\sqrt{1-t^2}\mathbf{k} \right\rangle dt$

23. $\int_3^6 \langle \sqrt{t-2}, t, 3 \rangle dt$

24. $\int_{1/4}^1 \left\langle \ln \sqrt{t}\mathbf{i} + e^{4t}\mathbf{j} - \frac{1}{t^{3/2}}\mathbf{k} \right\rangle dt$

25–26 Find the arc length of the curve over the given interval.

25. $\mathbf{r}(t) = \langle -3t^2, 6t, t^3 \rangle; \quad [0, 2]$

26. $\mathbf{r}(t) = \left\langle 2 \ln t, \frac{t^2}{4}, \sqrt{2t} \right\rangle; \quad [1, e^2]$

27–28 Reparametrize the given curve with respect to arc length.

27. $\mathbf{r}(t) = \langle 2-t, 1+4t, 3+2t \rangle$

28. $\mathbf{r}(t) = \langle 5 \sin t, 12t, 5 \cos t \rangle$

29–32 Find the unit tangent vector for the curve over the given interval.

29. $\mathbf{r}(t) = \left\langle \frac{t}{2}, \ln(\cos t), \frac{\sqrt{3}}{2}t \right\rangle; \quad 0 \leq t < \frac{\pi}{2}$

30. $\mathbf{r}(t) = \langle t^2, \cos 2t, \sin 2t \rangle; \quad t \geq 0$

31. $\mathbf{r}(t) = \langle \ln t, t^2, t \rangle; \quad t > 0$

32. $\mathbf{r}(t) = \langle \arctan t, \ln(t^2 + 1), t \rangle; \quad t > 0$

33–36 Find the unit normal and binormal vectors for the curve.

33. $\mathbf{r}(t) = \langle 2t, 2 \cos t, 2 \sin t \rangle$

34. $\mathbf{r}(t) = \langle \sin 3t, \cos 3t, 3t \rangle$

35. $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2t} \rangle$

36. $\mathbf{r}(t) = \langle 3t, -t, t^2 \rangle$

37–40 Using your work in Exercises 33–36, determine equations for the osculating, normal, and rectifying planes associated with the curve at the indicated point.

37. $\mathbf{r}(t) = \langle 2t, 2 \cos t, 2 \sin t \rangle; \quad t = 0$

38. $\mathbf{r}(t) = \langle \sin 3t, \cos 3t, 3t \rangle; \quad t = 0$

39. $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2t} \rangle; \quad t = 1$

40. $\mathbf{r}(t) = \langle 3t, -t, t^2 \rangle; \quad t = 2$

41–44 Use Exercises 37–40 to find a parametric description of the surface.

41. The ribbon of width $\frac{1}{2}$ centered on the curve
 $\mathbf{r}(t) = \langle 2t, 2 \cos t, 2 \sin t \rangle$

42. The ribbon of width 1 centered on the curve
 $\mathbf{r}(t) = \langle \sin 3t, \cos 3t, 3t \rangle$

43. The circular tube of radius 2 centered on the curve
 $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2t} \rangle$

44. The circular tube of radius 1 centered on the curve
 $\mathbf{r}(t) = \langle 3t, -t, t^2 \rangle$

45–48 Find the curvature and torsion functions for the given curve. (Suppose $t > 0$.)

45. $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{t}{\sqrt{2}} \right\rangle$

46. $\mathbf{r}(t) = \left\langle \frac{1}{t^2}, \frac{1}{t}, 1 \right\rangle$

47. $\mathbf{r}(t) = \left\langle t, \frac{1}{t}, \ln t \right\rangle$

48. $\mathbf{r}(t) = \left\langle t^2, t, \frac{1}{t} \right\rangle$

49. Find a formula for the curvature of the ellipse
 $\mathbf{r}(t) = \langle a \cos t, b \sin t, 0 \rangle$ ($a > b$) at the endpoints of its
a. major axis and b. minor axis.

50. Use Exercise 50 of Section 12.3 to provide a second solution for Exercise 49.

51–52 Use Exercise 50 of Section 12.3 to find the point(s) of maximum curvature for the curve.

51. $f(x) = \frac{1}{x}$

52. $f(x) = x^4$

53–54 Find the osculating circle of the graph of the equation at the indicated point. (See Exercises 60–68 in Section 12.3.)

53. $y = x^3; \quad x = -1$

54. $y = \tan x; \quad x = \frac{\pi}{4}$

55–58 Find the tangential and normal components of acceleration for the given position function.

55. $\mathbf{r}(t) = \langle 4t, \cos 2t, \sin 2t \rangle$

$$56. \mathbf{r}(t) = \left\langle \frac{1}{t}, 1, t^2 \right\rangle; \quad t > 0$$

$$57. \mathbf{r}(t) = \langle t \cos t, t \sin t, 1 \rangle; \quad 0 \leq t \leq \frac{\pi}{4}$$

$$58. \mathbf{r}(t) = \langle 1, 1 - \sin t, t - \cos t \rangle$$

59–60 Describe in terms of \mathbf{u} and \mathbf{u}_\perp the velocity and acceleration vectors of a particle with the given position function.

$$59. \mathbf{r}(t) = \left\langle t \cos \frac{t}{2}, t \sin \frac{t}{2} \right\rangle$$

$$60. \mathbf{r}(t) = \langle t^2 \cos 2t, t^2 \sin 2t \rangle$$

61. A projectile is launched from the ground with an initial speed of 129.33 feet per second at an angle of elevation of 60° from horizontal. Use the vector functions $\mathbf{v}(t)$ and $\mathbf{r}(t)$ to find the maximum altitude reached by the projectile as well as its range. (Ignore air resistance.)

62. The acceleration function of a moving object in \mathbb{R}^3 is $\mathbf{a}(t) = \langle t/2, 0, -t \rangle$. Find the velocity and position functions of this object if it starts at the point $\mathbf{r}(0) = \langle 3, -1, 5 \rangle$ with initial velocity $\mathbf{v}(0) = \langle 0, 2, 1 \rangle$.

63. Suppose a shell is shot from an artillery cannon with a muzzle velocity of 3000 ft/s and leaves the cannon 4 ft above the ground, at a 30° angle of elevation. What is the range of the cannon under these conditions? (By “range,” we mean the horizontal distance between the base of the launching pad and the point of impact. Assume level terrain and ignore air resistance.)

64. A projectile is launched from the point $(0, 8)$ of a coordinate system where unit length is measured in feet. The launching speed of the projectile is 192 feet per second and it is launched in the positive direction at an angle of elevation of 30° .

- Find a vector function that models the projectile's path.
- Find the maximum height attained by the projectile, its range, and the speed of impact.

(As customary, ignore air resistance.)

65. The banking of each of the curves on the historical oval track in Monza, Italy, is 30° , with a radius of 320 m. When used for racing, those banked curves were navigated by daring race drivers at speeds in excess of 150 mph (240 km/h). Find the vertical component of acceleration of a race car in the curve under the above conditions (assume a constant speed of exactly 150 mph).

66.* Using the result of Exercise 89 of Section 12.3, find the speed that the banking is able to support on the track of the previous exercise. In reality, much higher speeds were achieved in the curves of the track. Explain how that was possible.

67. A dive bomber, approaching its target at an angle of depression of 60° and at a speed of 600 kilometers per hour, releases its payload at an altitude of 500 meters. Use the units of meters and seconds to find **a.** a vector function modeling the path of the bomb, **b.** the horizontal distance traveled by the bomb after its release, and **c.** its speed of impact. (Convert your answer to part c. to kilometers per hour. You can suppose that the flight is taking place in the positive x -direction. Ignore air resistance.)

68. If a particle's acceleration along a path $\mathbf{r}(t)$ at time $t = t_0$ is $\mathbf{a}(t_0) = \langle 1, -2, -1 \rangle$, and its velocity $\mathbf{v}(t_0) = \langle 9, 21, -12 \rangle$, decide whether the particle's speed is increasing or decreasing at $t = t_0$.

69. Suppose a particle is moving along a path $\mathbf{r}(t)$ which is on the sphere of radius R , centered at the origin. Show that the velocity vector of the particle is tangent to the sphere at all times.

70.* Show that the moving particle of Exercise 69 satisfies $\mathbf{r}(t) \cdot \mathbf{r}''(t) = -|\mathbf{v}(t)|^2$.

71.* Suppose that an object in the three-dimensional coordinate system is moving along a path $\mathbf{r}(t)$ so that it is acted upon by a single, so-called central force; that is, if $\mathbf{F}(t)$ denotes the acting resultant force on the object, then $\mathbf{F}(t) = k\mathbf{r}(t)$, for some constant $k > 0$. Show that in this case, the motion is planar, that is, $\mathbf{r}(t)$ is contained in a plane.

72.* Show that if the vector function $\mathbf{r}(t)$ is *bounded*, that is, if $|\mathbf{r}(t)| \leq M$ for some positive constant M , then

$$\left| \int_a^b \mathbf{r}(t) dt \right| \leq M(b-a).$$

73. Find the curvature at $t = b$ for the Cornu spiral of Section 12.2, Exercise 44. What can you say about the relationship between arc length and curvature for this particular curve?

74. Show that the graph of the polar equation $r^2 = \cos^2 \theta - \sin^2 \theta$ has the property that its curvature is directly proportional to r .

75. Find the orbital speed of a satellite that is in a circular orbit, 300 kilometers above Earth's surface. Express your answer in kilometers per hour. (**Hint:** See Exercise 8 of Section 12.4.)

76. Ganymede, Jupiter's largest moon (discovered by Galileo Galilei in 1610), has an orbital period of approximately 7.15455296 earth days; its *periapsis* (distance from Jupiter upon closest approach) is approximately 1,069,008 kilometers, while its *apoapsis* (its greatest distance from Jupiter) is about 1,071,792 km. Use these data to estimate the mass of Jupiter. (**Hint:** See Exercise 10 of Section 12.4.)
77. Given that a day on Jupiter lasts a mere 9 hours and 55.5 minutes, use Exercise 76 along with Kepler's Third Law to estimate the necessary height above Jupiter's surface for a stationary satellite. (**Hint:** See Exercise 12 in Section 12.4. Approximate Jupiter's radius by 69,911 kilometers.)
78. The aphelion of Jupiter's orbit is 5.458104 AU (astronomical units), while its perihelion is 4.950429 AU. Use these along with Earth's orbital data to estimate the period of Jupiter in Earth years.
79. Use Exercise 78 to find the speeds of Jupiter when it is **a.** closest to and **b.** farthest from the sun. Express your answer in kilometers per second, then convert it to miles per hour. (**Hint:** See Exercise 23 in Section 12.4.)

Concept Check

80–89 Determine whether the given statement is true or false. In case of a false statement, explain or provide a counterexample.

80. $d\mathbf{T}/ds$ is perpendicular to \mathbf{T} .
81. If the graph of $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a straight line, then f , g , and h are linear polynomials.
82. The torsion of a curve satisfies the formula
- $$\tau = -\left(\frac{1}{|\mathbf{v}(t)|}\right) \frac{d\mathbf{B}}{dt} \cdot \mathbf{N}.$$
83. The tangential component of acceleration satisfies $a_T = \mathbf{T} \cdot \mathbf{a}$.
84. The normal component of acceleration satisfies $a_N = \mathbf{T} \times \mathbf{a}$.
85. If $f''(x) = 0$, then the curvature of the graph of $y = f(x)$ is constant.
86. If $\mathbf{r}(t)$ is a space curve with $\kappa = 0$, then $\mathbf{r}(t)$ can only "bend" in the direction of the unit binormal vector \mathbf{B} .

87. The magnitudes of the curvature and torsion of a space curve depend on the parametrization (i.e., "how fast the curve is being traced out,") unless the curve is parametrized with respect to arc length.
88. The maximum curvature of an ellipse occurs at the endpoints of its major axis.
89. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$

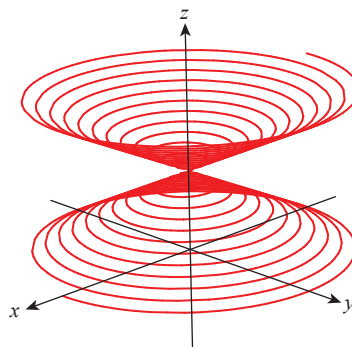
Chapter 12 Technology Exercises

90. **a.** Use a graphing utility to graph and explore the following curve.

$$\mathbf{r}(t) = \left\langle \frac{\sin 15t}{2}, (3 + \cos 15t)\sin t, (3 + \cos 15t)\cos t \right\rangle$$

(Such curves are nicknamed "slinky curves." Can you see why?)

- b.** It is possible to create "slinky curves" where the spiral is wound around a helix. Find a formula for such a curve. (**Hint:** A good starting point is appropriately modifying the formula in part a. Answers will vary.)
91. Assuming $|x|, |y| \leq 10$, z is nonnegative, and $0 \leq t \leq 10\pi$, use a graphing utility to find a formula for a vector function whose graph is as close as possible to the one displayed in the figure below. (Answers may vary.)



92. Write a program for a computer algebra system or programmable calculator that returns the equations for the osculating, normal, and rectifying planes associated with a given space curve at a specified point. Use your program to check the answers you have given to Exercises 37–40.